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STRUCTURAL ENGINEERING

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BY

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PREFACE

THE author does not profess in this work to treat the subject of Structural Engineering from a historical point of view, which he considers rather interferes with the continuity of the subject from a student's standpoint, and he may at once disclaim the possibility of introducing any very novel methods of presenting it. He considers, however, that certain aspects of the subject, for example, the methods of the equilibrium polygon and of the ellipse of stress, do not generally receive that attention to which their utility and reliability entitle them, and he has endeavoured to give them due prominence. He is also of the opinion that there is a want of a text book, suitable for students of Engineering, intermediate between what is generally called *Strength of Materials* and such specialised works as, for instance, Claxton Fidler's "Bridge Construction" or Merriman and Jacoby's "Roofs and Bridges." He has endeavoured to place before the student of the subject, in a consecutive and intelligible manner, the principal ideas and methods which underlie the investigations necessary in the design of structures. In most engineering problems it is not possible to make hypothetical assumptions, which, converted into mathematical equations, are a pure expression of the actual data of a problem of design, consequently mathematics have generally to be applied in detail to each part of a problem and the assumptions readjusted at each step to prevent the mathematical deductions diverging to any considerable extent from the actual conditions of the problem.

Whilst admitting that the success of the design of an engineering structure largely depends upon the sound judgment of the engineer, such judgment must be founded on the basis of the fundamental principles of statics—the ignoring of which would lead to waste or possible disaster.

This treatise may be taken as an attempt to set forth these fundamental principles.

The author has to thank Professor W. C. Unwin, F.R.S., for suggestions after reading over the proofs.

The author will be obliged by receiving notification of any mistakes that readers of the book may detect.

A. W. BRIGHTMORE.

LONDON, 1908.

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STRUCTURAL ENGINEERING

CHAPTER I

BENDING MOMENTS AND SHEARING FORCES

THE bending moment at any section of a structure, such as a beam or arch, for instance, is equal to the resultant of all the forces (including the reaction at the supports) to the right of the section multiplied by the perpendicular on this resultant force from the axis* of the structure at that section—it being understood that co-planar forces only are being considered; the sign is taken the same as for the moment of the force, *i.e.* positive if in the contra-clockwise direction. If the moment of the resultant force to the left of the section be taken, the bending moment would be equal to minus the moment of the resultant to the left.

The shearing force is the component of the resultant force to the right of the section resolved parallel to the section, and is taken as being positive if that component is acting upwards; if the resultant force to the left of the section be taken and resolved parallel to the section the shear would be positive if the component of that force acts downwards.

If the system of forces consist of vertical forces with a horizontal component of reaction at the support, as in the case of an arch or suspension bridge hinged at the supports, the resultant bending moment at any vertical plane is, by the definition, the algebraic sum of moments of the vertical forces, including, of course, the vertical component of the reaction, and of the horizontal component of the reaction. We may find the bending moment for the vertical forces separately from that for the horizontal forces, but the two component

* The axis of a structure is the line through the point in each cross-section, about which the moments of the stresses normal to the section are numerically equal. In case of vertical forces only acting, the moment of the acting forces on one side of a vertical section is the same about any point in it.

parts of the bending moment must be added algebraically to obtain the resultant bending moment.

It is obvious that the bending moment due to the horizontal force equals that force multiplied by the depth from the axis of the structure to the line joining the horizontal supporting hinges, and for an unvarying load it would only be necessary to suitably arrange that depth in order that the bending moment of the horizontal force should, at each section, be equal and opposite to the bending moment of the vertical forces, in that case the resultant bending moment would be equal to zero at every section; but the bending moment of the vertical forces would always have a certain value between the hinges, and be equal and opposite to that of the horizontal force.

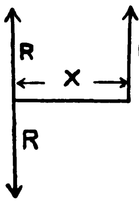


FIG. 1.

If a system of vertical forces in the same plane act on a structure at rest, such as a beam, their action on any vertical section is necessarily that due to the resultant of all the forces to the right or to the left of that section, as these must be equal and opposite, since the body is at rest; also the moment of all the forces on one side of that section must, for the same reason, be equal and opposite to the moment of all the forces on the other side. As from the above definition the bending moment at the section has the same sign as the moment of the forces on the right side, it is best to consider those on the right side, unless it is more convenient in any particular case to consider those on the left. We have, therefore, acting on the section a force = R, the resultant of all the forces to the right of the section at a distance of, say, X from the section. If we introduce two equal and opposite forces (Fig. 1) parallel and equal in magnitude to R, in the section, these will have no effect on the equilibrium, and these three forces will be equivalent to a force R acting in the section and a couple R × X acting on it. This force R in the section is the shearing force there, and the couple R × X is the bending moment there.

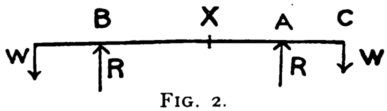


FIG. 2.

In case only vertical forces are acting, the bending moment at a section due to a load equals the moment about the section of its reaction at the right support, minus the moment of the part of the load to the right of the section. The bending

moment can only be zero at particular sections of the loaded beam, but it may be constant for a certain length of a beam. If, for instance, a beam is carried on two supports A and B equally distant from the two ends of the beam, and equal loads W act on the extremities of the beam, as in Fig. 2, then the bending moment on A B is constant, for the reactions at the supports will obviously be equal to the load at either end, and the bending moment at any point X between A and B equals $- W \times \overline{XC} + W \times \overline{XA} = - W \times \overline{AC}$, and is, therefore, constant between A and B.

Between the same points the shearing force will be zero because it equals $- W$ (at C) added to $+ W$ (at A).

A girder may be either simply supported at the ends of its span, the bending moments at these points being zero, in which case the loads carried by it would cause the girder to bend upwards at the ends; or, moments or couples may act on the ends tending to alter that inclination, *e.g.* if the ends are "fixed" so as to remain horizontal, couples would have to be introduced for this purpose, which would reduce the inclination at the ends to zero.

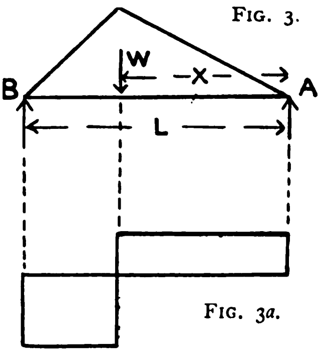


FIG. 3.

FIG. 3*a*.

I.—*When the Girder is simply supported at the ends*

If a single concentrated load W (Fig. 3) acts on a beam span L, at a distance X from A, the reaction at A $= R_A = W \times \frac{L - X}{L}$, and the reaction at B $= R_B = W \times \frac{X}{L}$, and the bending moment at a point distant x from A $= W \times \frac{L - X}{L} \times x$. In particular at the load the bending moment $= W \times \frac{L - X}{L} \times X$. To the left of the load the bending moment ceases to be represented by the same expression because the load also comes into the expression, and the bending moment therefore is

$$W \times \frac{L - X}{L} \times x - W(x - X) = \frac{W \times X \times (L - x)}{L}.$$

Thus, there is a discontinuity in the expression for the bending moment at the load in the case of a concentrated load.

The bending moment diagram is a line—straight, broken or curved, as the case may be—whose ordinate at any vertical section equals the bending moment there. It is obvious that the bending moment diagram for this load W is a triangle with its vertex on the line of action of the load, the ordinate there being

$$\frac{\mathrm{W} \times \mathrm{X} \times (\mathrm{L} - \mathrm{X})}{\mathrm{L}}.$$

If there is a number of loads on the span the bending moment diagram for each load may be drawn separately, and will, as shown above, each consist of a triangle with its vertex on the line of action of the load, the height of the ordinate there being equal to the load divided by the length of the span and multiplied by the length of each of the two segments into which the load divides the span. The sum of the ordinates of all these triangles at any section is the resultant bending moment at that section.

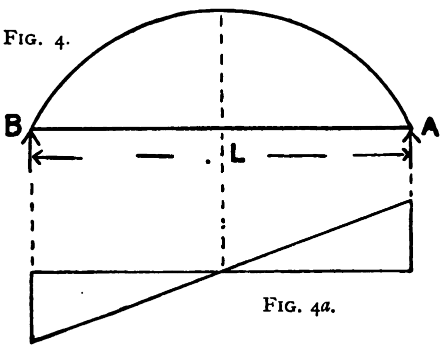


FIG. 4.

FIG. 4a.

If the span is covered by a uniform load of w tons per foot run (Fig. 4), the reactions are $\frac{w\,\mathrm{L}}{2}$ and the bending moment at a distance x from A

$$= \frac{w\,\mathrm{L}}{2} \times x - \frac{w\,x^2}{2} = \frac{w}{2}\,x\ (\mathrm{L} - x).$$

There is no discontinuity in this expression, it equals o at A and B, and it represents the equation to a parabola with its vertex on the vertical centre line.

It is often convenient to consider a uniform load as broken up into parts. It must be borne in mind that the ordinate of the bending moment diagram at any section is the moment of the total reaction at the right support due to all the loads, minus that of the portion of

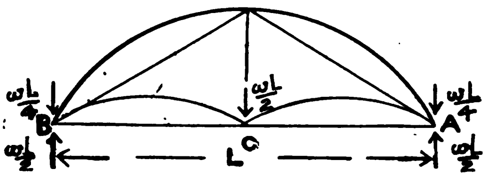


FIG. 5.

the loads to the right of the section. Supposing we divide the load into two equal parts at the centre; the parabola representing the bending moment diagram for the load on a span of half the length includes the moment of reactions $\frac{w\,L}{4}$ at each end of the loads; thus, if we want the moments of the uniform load only we must *subtract* from this parabola the moments of the reactions, or we must *add* the moments of loads equal and opposite to the reactions. In Fig. 5, loads equal and opposite to the reactions on two half spans are indicated, namely, $\frac{w\,L}{4}$ at A and B and $\frac{w\,L}{2}$ at C. Now, the beam is subject to the uniform load, and the reactions $\frac{w\,L}{2}$ at each end, and we have just seen that the two parabolas, each on half the span, plus the moments of loads $\frac{w\,L}{4}$ at A and B and $\frac{w\,L}{2}$ at C represent the moments of the load itself; thus, to complete the bending moment diagram, we have to add to the two parabolas on the halves of the span the moments due to loads $\frac{w\,L}{4}$ at A and B and $\frac{w\,L}{2}$ at C and the original reactions $\frac{w\,L}{2}$ at A and B. The loads at A and B counteract half these reactions and we have left,

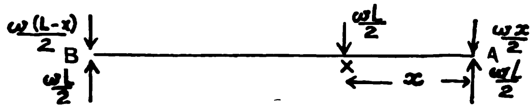


FIG. 6.

in addition to the small parabolas, the moments due to a load $\frac{w\,L}{2}$ at C and reactions $\frac{w\,L}{4}$ at A and B, the moments due to which are represented by the bending moment diagram for a load of $\frac{w\,L}{2}$ at C and its reactions, which is a triangle with vertex on the centre line and centre ordinate $= \frac{w\,L^2}{8}$.

Adding together the ordinates of the small parabolas and of this triangle, we get the original parabola shown by the heavy line. This shows that bending moment diagrams for parts of the load and their reactions may be combined to produce that for the whole load; *e.g.* we may substitute for the bending moment diagram for a uniform load covering the span, that for a

load equal $\frac{w\,L}{2}$ at the centre if we add the bending moment diagrams of a uniform load of the same intensity covering the half spans. Instead of dividing the span into two equal parts, divide it at any point X distance x from A, Fig. 6. Now two parabolas for the uniform load on spans x and $L - x$ represent the moments of such loads and their reactions, and to obtain the former only we must subtract the moments of their reactions or add the moments of loads equal and opposite to their reactions. The reactions for the uniform load or spans x and $L - x$ are $\frac{w\,x}{2}$ at A, $\frac{w\,(L - x)}{2}$ at B, and $\frac{w\,x}{2} + w\,\frac{(L - x)}{2} = \frac{w\,L}{2}$ at X. To obtain the bending moment at any section, in addition to the moment of the load, we have also the moment of the reactions $\frac{w\,L}{2}$, due to the original load, acting at A and B (Fig. 6). This leaves at A and B reactions $w\,\frac{(L - x)}{2}$ and $\frac{w\,x}{2}$ respectively and the load $\frac{w\,L}{2}$ at X, the bending moment diagram for which load and

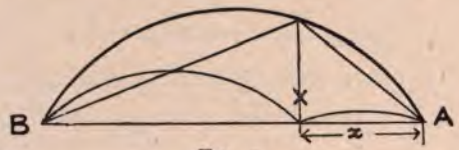


FIG. 7.

reactions is a triangle with vertex on the vertical through X, the ordinate there being $\frac{w}{2}\,(L - x)\,x$, which, it will be noticed, is also the ordinate for the uniform load at this section. Therefore the bending moment diagram is the sum of the bending moment diagrams for a concentrated load of $\frac{w\,L}{2}$ at X, and the parabolas representing the bending moment diagrams for the uniform load on the spans x and $L - x$, and the sum of the ordinates of the two, Fig. 7, gives the original parabola shown by the heavy line. It is obvious from this that the bending moment at any section for the uniform load covering the span is equal to that for a concentrated load equal to half the uniform load acting at that section. The analogy of a beam strained over one support to the beam supported at each end will be noticed shortly.

II.—*When the Girder is subjected to a couple at one end*

Take the case of a cantilever length l uniformly loaded (Fig 8), the bending moment for the uniform load alone is equal to that

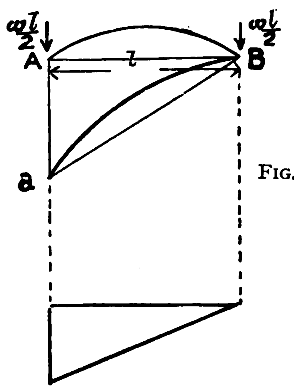


FIG. 8.

FIG. 8*a*.

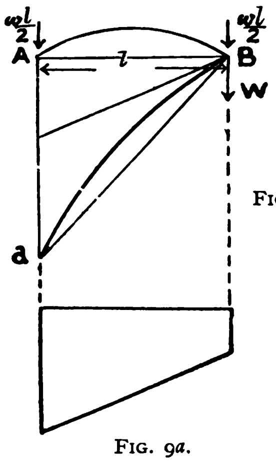


FIG. 9.

FIG. 9*a*.

for the load and its reactions (*i.e.*, the parabola for the uniform load on span l), minus the moment of the reactions $\frac{w\,l}{2}$ due to that load, or plus the moments of loads equal and opposite to the reactions. Introduce therefore loads equal $\frac{w\,l}{2}$ at A and B, and draw the parabola for a span A B due to the uniform load and its reactions. The moment at A due to $\frac{w\,l}{2}$ at B $= \frac{w\,l^2}{2}$ and is negative, and is the abutment moment; at any intermediate section the moment of this load is the ordinate of the line a B; the algebraic sum of the two sets of ordinates shown by the heavy line is the resultant bending moment diagram, and it is obvious by a comparison with Fig. 5 that it is half the parabola on base $2\,l$ with vertex at B.

If the cantilever has a load W at the end in addition to the uniform load, the negative moment at A $= \text{W}\,l + \frac{w\,l^2}{2}$; plotting this in Fig. 9 and joining a B, and plotting up vertically from a B the ordinates of the parabola on span l we obtain the resultant bending moment diagram.

Now take the case of a cantilever and shore span C A B uniformly loaded, and supported at A and B (Fig. 10). It

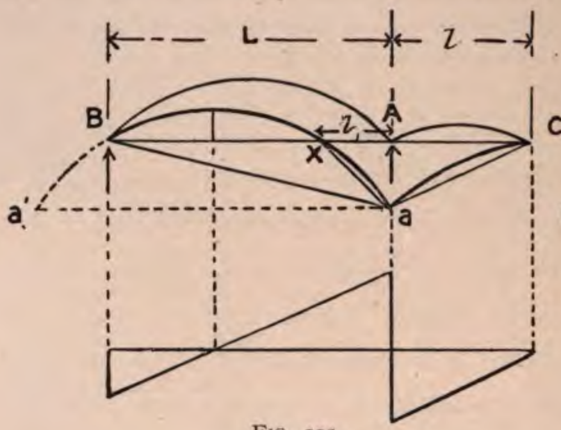


FIG. 10

FIG. 10a.

is obvious that, as in the last case, there is a negative moment at A due to the load on the span A C equal to $\frac{w\,l^2}{2}$; this exerts an uplift at B, diminishing the reaction there by an amount $= \frac{w\,l^2}{2\,L}$ and of course increasing that at A by the same amount. Therefore we have superimposed on the moment of the uniform load and its reactions on the span A B an additional positive reaction at A $= \frac{w\,l^2}{2\,L}$ and a negative reaction at B of the same amount; thus, as we move from A to the left the negative pier moment at A, equal $-\frac{w\,l^2}{2}$, is proportionately reduced by the positive moment of the additional reaction $\frac{w\,l^2}{2\,L}$ at A until at B the moment equals zero. Thus plotting $A\,a = \frac{w\,l^2}{2}$ and joining a to B and C, B a C is the diagram of bending moment due to the negative moment at the pier A, and on this we must superimpose the parabolas for the uniform load over the spans A C and A B. Let the ordinates of these parabolas be plotted upwards at each vertical from the negative moment diagram as base, and there results the curve B X a C, which is the resultant bending moment diagram. The point X, where the bending moment changes sign, is a point of

inflexion of the deflection curve. From Fig. 7 the parabola on A B is equivalent to the load $\frac{w\,l}{2}$ acting at X, whose moment is the triangle a X B, and the parabolas due to the uniform load on spans l_1 and $L - l_1$. Thus the curve a X B is part of the parabola on base $a\,a^1 = L + l_1$ with its vertex on the vertical through the centre of B X.

Now at X the moment of the uniform load $= \frac{w\,l_1}{2}\,(L - l_1)$ is equal to the negative bending moment at that point due to the negative reaction of $\frac{w\,l^2}{2\,L}$ at the pier B, which equals $-\frac{w\,l^2}{2\,L} \times (L - l_1)$.

$$\text{That is } \frac{w}{2}\,l_1\,(L - l_1) = \frac{w\,l^2}{2\,L}\,(L - l_1).$$

$$\therefore \quad l_1 = \frac{l^2}{L}$$

In Fig. 10 it will be noticed that we have a negative triangle of moments, and the two positive parabolas on spans l and L_1, whereas in Fig. 7 we have a positive triangle and the two positive parabolas on the two segments of the span.

The load on the pier A equals $w\,l + w\,l_1 + w\,\frac{(L - l_1)}{2}$ or $\frac{w}{2\,L}\,(L + l)^2$; and the load at B $= w\,\frac{(L - l_1)}{2} = w \times \frac{L^2 - l^2}{2\,L}$. If there is in addition a concentrated load on A B, the bending moment triangle on A B as base for this load and its reactions must be drawn, and its ordinates added algebraically to the ordinates of the bending moment diagram for the uniform load.

FIG. 11.

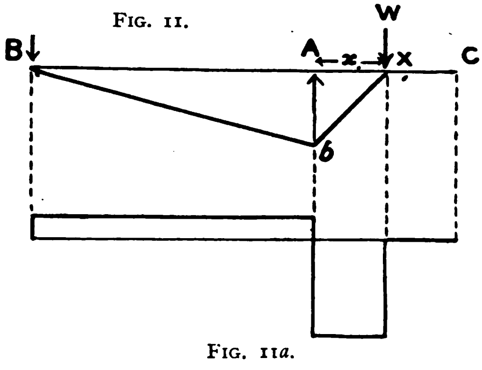


FIG. 11*a*.

If there is a load on A C say at X_1 distant x_1 from A (Fig. 11), plot the negative moment A b at A due to this load (in this case W x_1), and join b to B and X_1.

The diagram gives the resultant bending moment diagram due to that load the ordinates of which must be added to those previously obtained.

III.—*When the Girder is subjected to a couple at each end*

If the beam is strained over two piers A and B, as in Fig. 12, the lengths being as in that figure, if the load is uniform, plot the negative pier moments A a and B b at A and B equal to $\frac{w\,L_1^2}{2}$ and $\frac{w\,L_2^2}{2}$ respectively and join $a\,b$. Then the ordinate of $a\,b$ at any section is the negative moment at that section, because the negative moment A a at A tails out to zero at B, as shown by the line a B, and the negative moment B b at B tails out to zero at A, as shown by the line b A, and the sum of the ordinates to A b and B a from A B equals the ordinate of $a\,b$. From the line D b a C plot upwards at each section the ordinate of the parabola for the uniform load on spans L_2, L, and L_1 and the resultant bending moment diagram is evidently obtained.

FIG. 12.

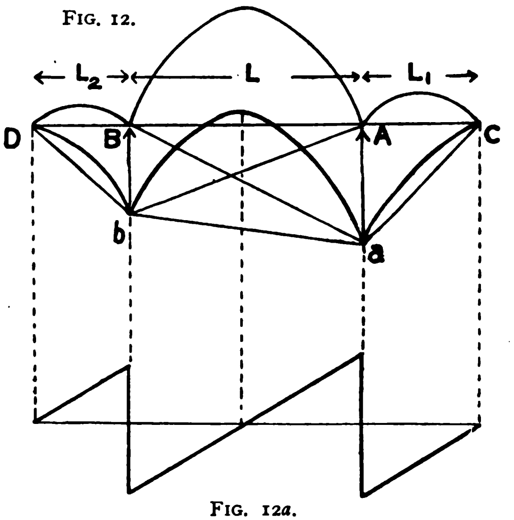


FIG. 12*a*.

FIG. 13.

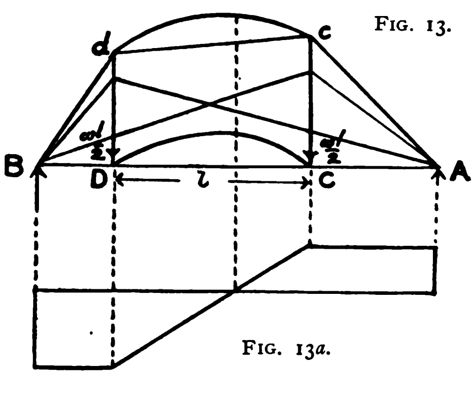


FIG. 13*a*.

n case a uniform load covers only length l forming part span length L (Fig. 13), supported at both ends, introduce ; at the ends of the uniform load equal and opposite to eactions of the uniform load on a span equal to its length. v the bending moment diagrams for these concentrated loads iown in Fig. 13, and add their ordinates, which gives the ram A c d B; on c d plot the ordinates of the parabola for iniform load on span C D, and the bending moment diagram viously complete.

Iitherto it will be noticed that the resultant bending moment ram has been uced by com-ng *bending ent diagrams* arts of the load their corres-ling reactions, there is an ely distinct iod of arriving he bending ient diagram combining the *ents* of all the ig forces in-ng the *total* :tions. This iod is very il when, in-l of a single entrated load uniform load ring the span irt of it, the load consists of a series of concentrated loads or without a portion of the load uniformly distributed.

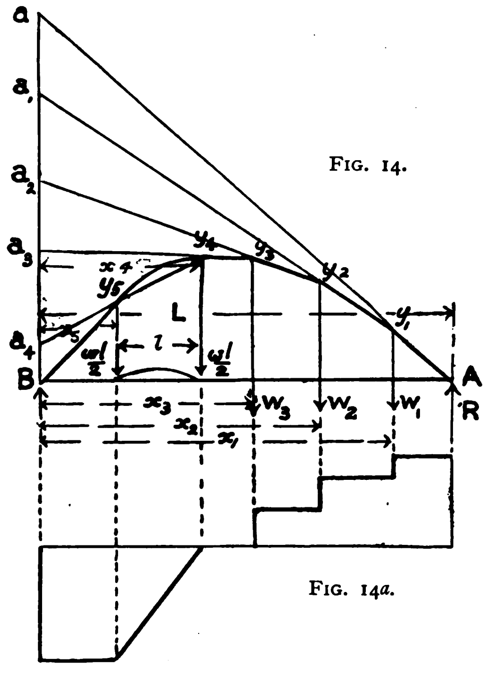


FIG. 14.

FIG. 14*a*.

'o show how this method is applied, take first the case 14) of a girder supported at the ends and loaded with : concentrated loads W_1, W_2, W_3 and a uniform load length l; titute for the uniform load a load $\frac{w\,l}{2}$ at each end, equal and site to the reactions of the uniform load on a span l, and y, after drawing the bending moment diagram for these

concentrated loads, add the ordinates of the parabola on a span of that length to the ordinates of that portion of the diagram. First find R $\times$ L by taking moments about B; this is not really necessary, but it is a check on the work, and more convenient. At B plot upwards, because it is a positive moment, B a equal to R $\times$ L, and join A a. Then any ordinate of A a will represent the moment of R at that ordinate. From a plot downwards, $a\,a_1$ equal to $W_1\,x_1$, and join a_1 to the point y_1 where the line A a cuts the line of action of W_1. Then any ordinate of the triangle $a\,y_1\,a_1$ is the moment of the load W_1 about that ordinate. From a_1 plot downwards $a_1\,a_2$ equal to $W_2\,x_2$ and join a_2 to the point y_2, where $y_1\,a_1$ cuts the line of action of W_2. Similarly draw $a_2\,a_3$ equal to $W_3\,x_3$, $a_3\,a_4$ equal to $\frac{w\,l}{2} \times x_4$ and a_4 B equal to $\frac{w\,l}{2} \times x_5$, which must necessarily come back to B, since the moment of the downward loads equals the moment of the upward reaction; join a_3 to y_3, a_4 to y_4, and B to y_5. Above the line $y_4\,y_5$ plot the ordinates of the parabola on base l. Then A $y_1\,y_2\,y_3\,y_4\,y_5$ B is the bending moment diagram, because its ordinate at any section equals the moment of the reaction at A about it, minus the moments about it of all the loads between A and the section. It will be seen when considering the equilibrium polygon that this construction is identical with that for this polygon for the same loads.

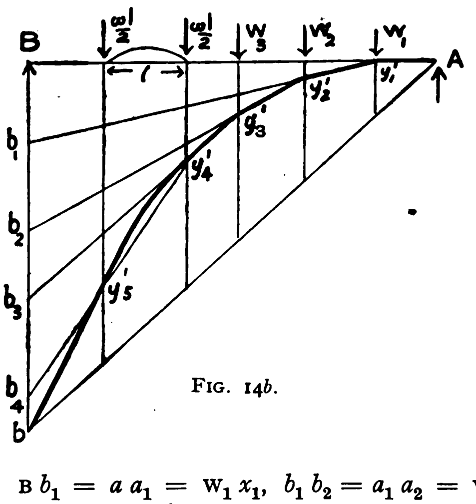


FIG. 14b.

It is sometimes convenient to draw the diagram of moments entirely below the line A B (Fig. 14b). For this purpose plot B b vertically downwards equal and opposite to the moment of the reaction at A. Draw B $b_1 = a\,a_1 = W_1\,x_1$, $b_1\,b_2 = a_1\,a_2 = W_2\,x_2$, etc. Join b_1 to y_1^1, b_2 to y_2^1, etc., and join A b. Then A $y_1^1\,y_2^1\,y_3^1\,y_4^1\,y_5^1\,b$ is the bending moment diagram referred to base A b, its

ordinates being the same as for the diagram drawn above A B.

Next take the case of the cantilever arm and shore span as before, but let it now be loaded with a uniform load from A to B with concentrated loads W and W_1 at C and D respectively (Fig. 15). At A and B introduce loads $\frac{w\,L}{2}$ equal and opposite to the reactions of the uniform load on a span L. Find R × L the moment of the reaction at A by taking moments about B. From B plot B c equal to W × (L + l) vertically downwards, and join C c. From c plot $c\,a$ equal to R × L vertically upwards, and join a to a^1, the point of intersection of C c with the line of action of R. From a plot $a\,d$ equal to $\frac{w\,L}{2}$ × L vertically downwards, and join $d\,a^1$. From d plot d B equal $W_1 \times x_1$ vertically downwards, and join B to d^1 the point of intersection of $a^1\,d$ with the line of action of W_1. This comes back to B, because the moment of the reaction at A about B is equal to the moments of all the downward forces. To complete the diagram, the ordinates of the parabola for the uniform load on the span A B must be plotted upwards from the broken line $a^1\,d^1$ B. Now the broken line C $a^1\,d^1$ B is the bending moment diagram for the loads W, $\frac{w\,L}{2}$,

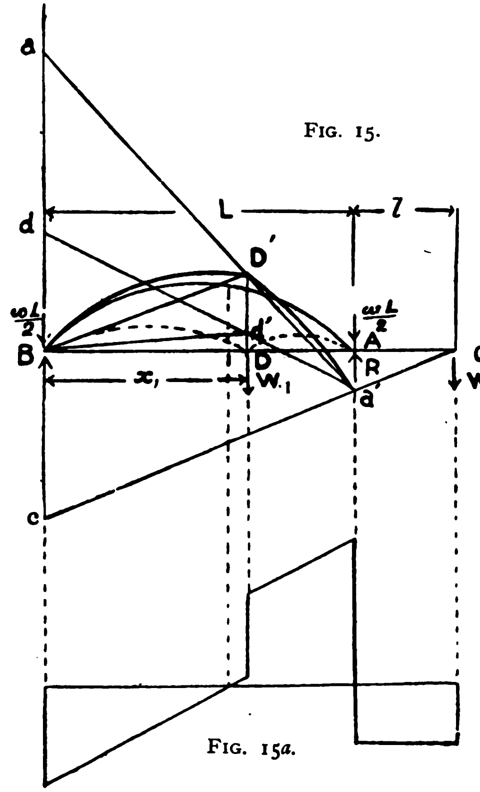


FIG. 15.

FIG. 15a.

W_1 and $\frac{w\,L}{2}$, because any ordinate is the difference of the positive and negative moments of the forces to the right of that ordinate, and if instead of $\frac{w\,L}{2}$ at A and B we have the uniform load of intensity w over the span L, we must add, as mentioned above, the ordinates of that parabola at each corresponding section. Therefore the curved diagram C a^1 D^1 B is the resultant bending moment diagram.

Instead of taking a load $\frac{w\,L}{2}$ at either end of the uniform load to obtain a skeleton diagram and then adding the ordinates of the parabola on A B as base, a load $\frac{w\,L}{2}$ may be taken as acting at D in addition to the load W_1 there, then C a^1 D^1 B is the skeleton diagram, and on a^1 D^1 and D^1 B must be plotted the vertical ordinates of the parabolas on A D and D B. The reasons for this were explained in connection with Fig. 7.

This method can be employed for any distribution of load provided the reactions can be found, and the application of it in different cases only varies in detail, and is of great practical utility.

The methods first treated serve to throw light on the combining of bending moment diagrams, and the effect of a moment over a pier, which sometimes are not easily grasped.

The shearing-force diagrams are drawn underneath each of the bending moment diagrams given.

Fig. 3*a* gives the shearing-force diagram for a single concentrated load on a span. At A the shear equals the reaction there, and remains constant until the load is reached, when it is diminished by the amount of the load and therefore becomes negative, and equal and opposite to the reaction at B, remaining constant up to that point.

Fig. 4*a* is the shearing-force diagram for a uniform load on a span. At A the shear equals the reaction there, and is gradually diminished by the downward load, as the point travels to the left, becoming zero at the centre, and then a gradually increasing negative quantity till at B it is equal and opposite to the reaction.

Fig. 8*a* is the shearing-force diagram for a uniform load on a cantilever; the shear starts equal to zero at B, and is a gradually increasing negative quantity until at A it is equal to the whole load.

In Fig. 9*a*, where there is a load W at the extremity B of the cantilever, the shearing force at B is negative and equal to W, and as A is approached it is a gradually increasing negative quantity, until at A it equals W plus the uniform load.

Fig. 10*a* is the shearing-force diagram for a uniform load on the cantilever arm and shore span. At C it is zero, and is a gradually increasing negative quantity to A, where it equals the load on the cantilever arm, then it suddenly increases to a positive quantity by the amount of the reaction at A, and gradually decreases again until at B it is equal and opposite to the reaction at that point. It passes through zero at A and on the vertical through the middle point of B X, where the bending moment is a maximum. This is necessarily the case, as the shearing force is the differential coefficient of the bending moment. It will be noticed that the sloping lines in the shearing-force diagram are of constant inclination when the intensity of the uniform load is constant.

In Fig. 11*a* there is a single load on the cantilever arm, the shearing force is therefore zero from C up to the load, where it becomes equal to the load and negative, and remains constant up to the pier A; it is then increased by the reaction to a positive quantity, and remains constant up to B, where it is equal and opposite to the negative reaction caused at that pier by the pier moment at B.

In Fig. 12*a*, where the girder is strained over two piers and is uniformly loaded, the shearing force at C is zero, and is a gradually increasing negative quantity up to the pier A, where it is increased by the reaction at that pier to a positive quantity; it then gradually decreases, passing through zero, where the bending moment is a maximum, and then becoming negative. When the pier B is reached it is suddenly increased to a positive value by the reaction there, and then gradually decreases to D, where it again equals zero. The three sloping lines are parallel, because the load is of uniform intensity.

In Fig. 13*a* the girder has a uniform load covering part of its length only; at A the shearing force is positive and equal to the reaction, and continues constant until the load is reached; it then gradually diminishes, passing through zero, where the bending moment is a maximum, and at the end of the load having a negative value equal and opposite to the reaction at B.

In Fig. 14*a*, where the girder is subjected to three concentrated loads and a uniform load of length l, the shearing force

at A equals the reaction there; it steps down at each of the concentrated loads, remaining constant between them; it slopes down for the length of the uniform load, and to the left of it becomes equal and opposite to the reaction at B.

In Fig. 15*a*, where the cantilever arm has a load at its extremity and the shore span is covered by a uniform load with a concentrated load in addition, the shearing force at C is negative and equal to the load there, and remains constant to A, where it is suddenly increased by the reaction, and then gradually diminishes till the concentrated load is reached, when it is decreased suddenly by that amount, and continues to diminish, passing through zero, where the bending moment is a maximum, and becoming negative, till at B it is equal and opposite to the reaction there.

BENDING MOMENT FOR TWO MOVING LOADS ON A GIRDER

If a load W (Fig. 16) is moving over a girder of span L, and is at X, a distance x from the end A, the reaction at $B = R_W = \frac{x}{L} \times W$, and therefore the bending moment at $X = M_W = \frac{W\,x\,(L - x)}{L}$ which is the equation to a parabola with its vertex on the vertical at the centre of the span. Thus for the load in any position the ordinate to this parabola gives the bending moment at the load. The bending moment is a maximum at any point when the load is at that point, as for any other position of the load the bending moment at the point is the ordinate to a straight line drawn from A or B to the point of intersection of the parabola by the vertical through the load.

Suppose a second load W_1 is connected to the first at a constant distance x_1 from it, and suppose that W_1 is less than W. W_1 will not come on to the bridge till x is greater than x_1, when $x > x_1$ the reaction at B due to $W_1 = R_{W_1} = W_1 \frac{x - x_1}{L}$, and therefore the bending moment at X due to $W_1 = M_{W_1} = W_1 \frac{(x - x_1)}{L} \times (L - x)$. Or if we write $x - x_1 = z$, $M_{W_1} = W_1\, z \frac{(L - x_1 - z)}{L}$.

Now this expression is the ordinate of a parabola of base equal to $(L - x_1)$ at the distance z from the end. If this parabola be plotted as in Fig. 16, the ordinate distant z or $x - x_1$ from its extremity nearer A is distant x from A, therefore the ordinates M_W and M_{W_1} coincide, and the sum of the two is the bending

moment M_x at the section considered—*i.e.*, at the position of W. It is obvious that the sum of the two ordinates for a section on the left of the centre is greater than for a section on the right of the centre line when the loads approach from the right

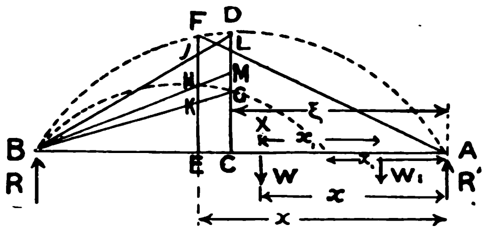


FIG. 16.

side as in the figure, because the vertex of the smaller parabola is to the left of the centre. In other words, the bending moment is greater at the greater load when $x > \frac{L}{2}$ than when $x < \frac{L}{2}$—*i.e.*, when the smaller load is on the longer segment into which the greater load divides the bridge. When x is less than $\frac{L}{2}$ the moment at x is greater if W_1 is to the left of W, which is equivalent to the load coming on from the left and measuring x from B instead of A.

Now—

$$M_x = M_W + M_{W_1} = \frac{W\,x\,(L - x)}{L} + \frac{W_1\,(x - x_1)\,(L - x)}{L} \quad . \quad . \quad (1)$$

$$= \frac{L - x}{L}\ \{(W + W_1)\,x - W_1\,x_1\} \quad . \quad . \quad (2)$$

This shows that the bending moment is greater if $W > W_1$, so that the smaller weight comes in the term which is subtracted in this expression. By differentiating (2) and equating to zero, M_x is a maximum when $(L-x)\,(W + W_1) - (W + W_1)\,x + W_1\,x_1 = 0$,

reducing, $2\,x\ (W + W_1) - L\,(W + W_1) - W_1\,x_1 = 0$,

$$i.e., \text{ when } x - \frac{L}{2} = \tfrac{1}{2}\,\frac{W_1\,x_1}{W + W_1}.$$

Now $\frac{W_1\,x_1}{W + W_1}$ is the distance of the centre of gravity of the two loads from W, therefore, for the maximum value of M_x, W must be beyond the centre a distance equal to half the distance of the centre of gravity of the loads from W.

The maximum M_x

$$= \frac{1}{2L}\left(L - \frac{W_1}{W + W_1} x_1\right) \times \left\{\frac{W + W_1}{2} \times \left(L + \frac{W_1}{W + W_1} x_1\right) - W_1 x_1\right\}$$

$$= \frac{1}{2L}\left(L - \frac{W_1}{W + W_1} x_1\right) \times \left(L \times \frac{W + W_1}{2} + \frac{W_1 x_1}{2} - W_1 x_1\right)$$

$$= \frac{1}{2L}\left(L - \frac{W_1 x_1}{W + W_1}\right) \times \left(L - \frac{W_1 x_1}{W + W_1}\right) \frac{W + W_1}{2}$$

$$= \frac{W + W_1}{4L}\left(L - \frac{W_1 x_1}{W + W_1}\right)^2$$

which is less than $\frac{(W + W_1) \times L}{4}$, the value of the bending moment if both loads are at the centre.

This proposition may be extended to any number of loads, each additional load adding another term to equation (1), of exactly the same type as the second term, but instead of W_1 and x_1 we have the additional weight and its distance from W.

It is not obvious without further inquiry that the bending moment at X is greatest when W is at X. If we consider the load W (Fig. 16) to be at C, W_1 being to the right of W, it requires very little consideration to see that at a point to the left of the load, say at E, the bending moment is less than when the load is at E. For with W at C the bending moment at E is EJ + EK (because BD and BG are lines of the bending moment diagrams of W and W_1 respectively), which are obviously less than EH + EF, the bending moment when the load is at E. But if the load W is at E, and we consider the bending moment at a point C between W and W_1, then the bending moment at C = CL + CM (because AF and BM are lines of the bending moment diagrams of W and W_1 respectively), and it is not obvious that CD + CG > CL + CM. To prove that this is the case :—

Call ξ the distance of the section CD from A, the load being at E at a distance x from A, then :—

The bending moment at C with the load W at E

$$= M_{CE} = R_W (L - \xi) - W(x - \xi) + R_{W_1} (L - \xi)$$

$$= \frac{L - \xi}{L} \times \{W x + W_1 (x - x_1)\} - W(x - \xi).$$

From equation (2) the bending moment at C with the load W at $C = M_C = \frac{L - \xi}{L} \{(W + W_1)\, \xi - W_1 x_1\}$.

Now $M_{CE} > M_C$ if

$$\frac{L-\xi}{L}\{W x + W_1(x - x_1)\} - W(x - \xi) > \frac{L-\xi}{L}\{(W + W_1)\xi - W_1 x_1\}.$$

hat is if—

$$\frac{L-\xi}{L} \times (W + W_1)x - W(x - \xi) > \frac{L-\xi}{L} \times (W + W_1)\xi,$$

r— $(L - \xi)(W + W_1)(x - \xi) > W L(x - \xi)$

ancelling— $(L - \xi)(W + W_1) > W L$

$$L W_1 > \xi (W + W_1)$$

$$\text{or if } \xi < \frac{W_1}{W + W_1} \times L \text{ which is } < \frac{L}{2}.$$

Now we have seen that M_C is a maximum when $\xi > \frac{L}{2}$, therefore the maximum bending moment at C occurs when the load is at that point.

If there are three loads on a beam, the third one W_2 being less than either of the other two, to find when the bending moment would be greater at the second load than at the first—

First let the first load W *be at* X, W_1 being at a distance

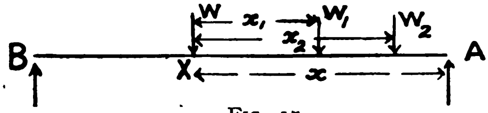


FIG. 17.

of x_1 to the right of W, and W_2 a distance of x_2 to the right of W (Fig. 17).

Then from equation (I)—

$$M_x = \frac{W x(L - x)}{L} + \frac{W_1(x - x_1)(L - x)}{L} + \frac{W_2(x - x_2)(L - x)}{L}$$

$$= \frac{L - x}{L}\{(W + W_1 + W_2)x - W_1 x_1 - W_2 x_2\}$$

Next let the second load be at X (Fig. 17).

Then— $$M_{x_1} = \frac{W(x + x_1)(L - x)}{L} + \frac{W_1 x(L - x)}{L} + \frac{W_2(x + x_1 - x_2)(L - x)}{L} - W x_1$$

$$= \frac{L - x}{L}\{W(x + x_1) + W_1 x + W_2(x + x_1 - x_2)\} - W x_1$$

$$= \frac{L - x}{L}\{(W + W_1 + W_2)x + (W + W_2)x_1 - W_2 x_2\} - W x_1.$$

Therefore, $M_{x_1} > M_x$, if

$$\frac{L - x}{L} \times (W + W_2)\, x_1 - W x_1 > - \frac{L - x}{L} \times W_1 x_1$$

simplifying, if $\frac{L - x}{L} (W + W_1 + W_2)\, x_1 > W x_1$

or if $$L - x > \frac{W}{W + W_1 + W_2} \times L.$$

i.e. if $$x < \frac{W_1 + W_2}{W + W_1 + W_2} \times L.$$

Next consider the maximum shearing force for the case of two loads on the span.

The positive shear between W_1 and $W = R_A - W_1$, where R_A is the reaction due to W and W_1 at A.

$$= \frac{W(L - x)}{L} + \frac{W_1(L - x + x_1)}{L} - W_1$$

$$= W - \frac{W x}{L} + W_1 - \frac{W_1 x}{L} + \frac{W_1 x_1}{L} - W_1$$

$$= W - \frac{x}{L}(W + W_1) + \frac{W_1 x_1}{L}.$$

This is a maximum when W is greater than and precedes W_1, as placed in Fig. 16, if $x = 0$ when its value equals W, since W_1 is not then on the span.

The maximum negative shear is to the left of W and $= - R_B$, where R_B is the reaction at B due to W and W_1.

$$- R_B = - \frac{W x}{L} - W_1 \frac{(x - x_1)}{L}$$

$$= - \frac{x}{L} (W + W_1) + \frac{W_1 x_1}{L}.$$

This has its greatest value when $x = L$, when it equals $- W - W_1 \left(1 - \frac{x_1}{L}\right)$. Thus the greatest positive value of the shear would occur if the smaller load precedes the larger one, when it would be $W + W_1 \left(1 - \frac{x_1}{L}\right)$.

Let a uniform load of intensity w advance over a span length L (Fig. 18). Let the distance of the head of the load from A be x. The positive shear at the head of the load, if the load advances from the left, equals the reaction at $A = \frac{w\,(L - x)^2}{2\,L}$, which has its maximum value when $x = 0$, when it equals $\frac{w\,L}{2}$. The negative shear if the load advances from the

right equals the reaction at B $= -\dfrac{w\,x^2}{2\,\text{L}}$, which is a maximum when $x = \text{L}$, when it equals $-\dfrac{w\,\text{L}}{2}$. In Fig. 18 the curves a B, A b represent the maximum value of the shear at each section for the load advancing from the left and from the right respectively. If there is, in addition, a uniform dead load on

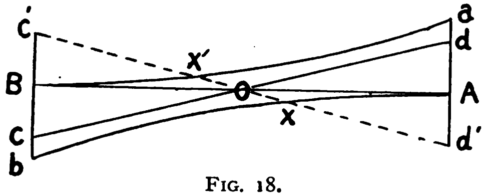


FIG. 18.

the bridge, the shearing-force diagram of which is c O d, if this line be reversed with respect to A B, shown by the dotted line c^1 O d^1, it will be seen that between the section at X and O the negative shear due to the live load advancing from the right will exceed the positive shear due to the dead load; similarly between O and the section at X^1 for the live load advancing from the left. Thus between the sections X and X^1 the shearing stress will change sign due to the live load, which means that the braces between these sections which are in tension due to the dead load would be put into compression, unless counterbraced, for the positions of the live load indicated in Fig. 18 between the sections X and X^1.

CHAPTER II

THE EQUILIBRIUM POLYGON

THE equilibrium polygon has a variety of uses, and is invaluable for finding the position of the resultant of a series of forces acting on a structure, and the reactions at the supports or at hinges, particularly when the acting forces are not parallel. In the case of vertical forces it can be used not only to find the bending moment at any vertical section, but also to determine the maximum shearing forces due to a moving load. Its use has been partly masked by the name "link" polygon,

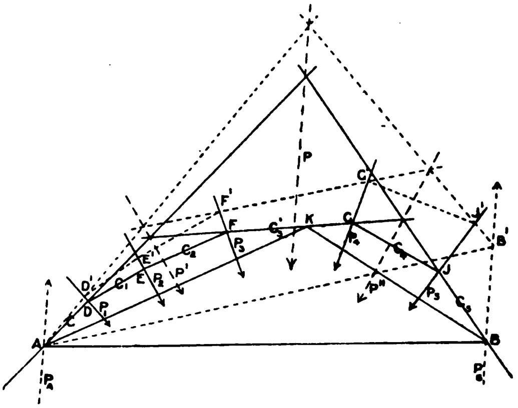


FIG. 19.

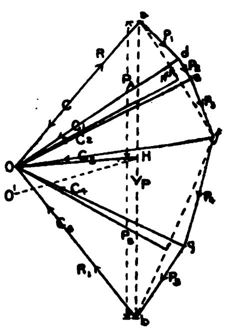


FIG. 20.

by which it is sometimes known, which implies a much more restricted application than it is capable of. Suppose P_1, P_2, P_3, P_4, and P_5 to be a series of forces acting on a structure in the plane of the paper in the positions indicated in Fig. 19. Let the force diagram be drawn in Fig. 20 to any convenient scale; starting at a, P_1 is drawn parallel to its actual direction, the length of the line representing its magnitude being to the scale chosen; P_2 is drawn on the same scale from the end of P_1, parallel to its actual direction, and so on, the end of the line

representing P_5 in magnitude and direction being at *b*. The line *a b* would then represent the resultant P of these five forces in magnitude and direction, and if *a* be joined to *f* and *f* to *b*, *a f* is the resultant of the three forces P_1, P_2, and P_3, and *f b* the resultant of P_4 and P_5. In Fig. 20 let any pole O be taken and joined to the extremities of the lines representing each of the forces, and call the forces represented by these rays in magnitude and direction, acting towards O, C, C_1, C_2, C_3, C_4, and C_5, as shown in Fig. 20. Then C and $-C_1$ are equivalent to P_1, C_1 and $-C_2$ are equivalent to P_2, and so on till we come to P_5, which is equivalent to C_4 and $-C_5$. Adding these components, it follows that C and $-C_5$ are equivalent to the five forces P_1 to P_5 or their resultant, as is obvious from the figures; or generally :—

(*a*) The force represented by the ray at the beginning of the first force considered, combined with that represented by the ray reversed at the end of the last force, is equal to the resultant of all the intermediate forces.

If we write $R = -C$ and $R_1 = +C_5$, and call R and R_1 the reactions, they close the force polygon—*i.e.* the structure is in equilibrium under the resultant of the applied forces, R and R_1. The following general statement can now be made :—

(*b*) The force represented by any ray in magnitude and direction, and acting away from O (or to the right in the present instance), is the resultant of the reaction R and of all the forces between R and the ray in question. It follows that the force represented by this ray, but acting in the opposite direction—*i.e.* towards O—is the resultant of the reaction R_1 and of all the forces between R_1 and the ray, since R and R_1 close the force polygon.

In Fig. 19, from a point D in the line of action of P_1, draw lines parallel to C and $-C_1$, from the point E where the latter meets the line of action of P_2 draw a line parallel to $-C_2$, and so on, till from the point J on the line of action of P_5 a line is drawn parallel to $-C_5$. Thus a force C in A D and a force $-C_1$ in D E, acting away from D, are equivalent to P_1, a force C_1 in E D and a force $-C_2$ in E F, acting away from E, are equivalent to P_2, and so on, and a force of C_4 in J G and a force of $-C_5$ in J B are equivalent to P_5. Thus, instead of the forces P_1 to P_5, we may substitute a force C in A D acting from D to A, a force C_1 in D E, acting from D and E towards its centre, C_2 in E F, C_3 in F G, C_4 in G J, acting from G and J towards its centre,

and C_5 in J B acting from J to B. Then A D E F G J B is an equilibrium polygon for the forces, because equal and opposite forces to those just enumerated would be in equilibrium with the original forces. Thus the equilibrium polygon is the shape of a structure such as a series of hinged links that would be in equilibrium under the acting forces and the reactions as given in the force diagram.

As the figure is drawn the equilibrium polygon is above A B, but if O were taken to the right of the forces in Fig. 20 instead of to the left, the equilibrium polygon would be situated below A B, and the direction of the substituted forces along the lines of the polygon would be opposite to the above.

It follows from the conclusion (*a*) above that: I.—If two lines in the equilibrium polygon be produced, the resultant of all the forces between the corresponding rays in the force diagram passes through their intersection, and is of course parallel to the resultant in the force diagram. For example, A D and B J produced intersect on the resultant P of the five forces P_1 to P_5, which is equal and parallel to $a\,b$ in Fig. 20. Similarly, A D and G F intersects on the resultant P^1 of P_1, P_2, and P_3, which is equal and parallel to $a\,f$ in Fig. 20, and F G and B J produced intersect on P'' the resultant of P_4 and P_5, which is equal and parallel to $f\,b$ in Fig. 20—because in each case the lines produced are the two components of the resultant.

II.—It follows from the conclusion (*b*) above that, if a section be taken cutting the equilibrium polygon, the force acting to the right* along the line intersected would be the resultant of the reaction R and all the forces to the left of the section; and the equal and opposite force acting along it to the left* is necessarily therefore the resultant of the reaction R_1 and all the forces to the right of the section. If the equilibrium polygon or any part of it is a continuous curve, the tangent to such curve at any point would, of course, be the direction of the resultant force. Now join the point A in A D to the point B on J B, where A and B are the points of support, and draw a line parallel to this in the force diagram to intersect $a\,b$ in H; then the forces represented in magnitude and direction by b H and H O are equivalent to R_1, and the forces represented in magnitude and direction by O H and H a are equivalent to R—*i.e.* b H and H a are the components of the reactions R_1 and R parallel

* With the pole on the right of the forces in Fig. 20, these directions would, of course, be reversed.

to the resultant when H O and O H are their equal and opposite components along the closing line A B. Similarly, if A be joined to any point K on F G and O H^1 be drawn parallel to A K to intersect *a f* in H^1, then *f* H^1 and H^1 O are equivalent to C_3 and O H^1 and H^1 *a* are equivalent to R—*i.e.* *f* H^1 and H^1 *a* are the components of C_3 and R respectively parallel to the resultant *a f* when H^1 O and O H^1 are their equal and opposite components along the closing line A K. Similarly for the closing line K B. Therefore :—

III.—If any closing line be drawn by joining two points on any two lines of the equilibrium polygon, and a line be drawn from O in the force diagram parallel to this closing line, this line represents in magnitude and direction the equal and opposite components of the reactions in the direction of the closing line, while the two segments into which the resultant is thus divided are their components parallel to the resultant. *Note.*—The reactions are the forces acting along the lines of the equilibrium polygon towards their intersection, on which the extremities of the closing line lie, in accordance with their definition on page 23.

In Figs. 19 and 20, R acting at A and R_1 acting at B balance the forces P_1, P_2, P_3, P_4, P_5 ; now R is the resultant of O H and P_A and R_1 is the resultant of P_B and H O, where P_A and P_B are parallel to and together equal to the resultant, we may therefore resolve R at A into P_A and O H, and R_1 at B into H O and P_B; thus P_A and O H at A and H O and P_B at B balance the system of acting forces. If a member connects A to B subject to a stress H equal to O H and H O, then there would only be required the forces P_A at A and P_B at B to balance the system. It is sometimes more convenient to regard P_A and P_B as two of the acting forces which close the force polygon, then in completing the equilibrium polygon from B lines are drawn parallel to C_5 and H O, and from A lines parallel to O H and $-$ C, thus all the components parallel to the rays are balanced, and we have left the equal and opposite forces along A B, which also balance.

The values of P_A and P_B are found by drawing from O a line parallel to the closing line A B, where A and B are the points of support, but they are dependent simply on the actual position of the resultant relatively to A and B, for P_B may be found by taking moments about A, when P_B multiplied by its perpendicular distance from A equals the resultant multiplied by its perpendicular distance from the same point. It therefore follows that if a

different pole o^1 be taken, and lines be drawn through A and B parallel to the resultant, which will of course represent the lines of action of P_A and P_B, and the equilibrium polygon for the pole o^1 be drawn starting at A, it will finish up at some point B^1 on the line of action of P_B. A line drawn from o^1 parallel to the closing line $A\,B^1$ will intersect it at H again, because it divides $a\,b$ into the two portions P_A and P_B; therefore:—

IV.—If a structure is supported by two hinges or equivalent joints, and the direction of the resultant of the acting forces other than the reactions be found; by drawing lines through the two hinges parallel to this resultant and drawing an equilibrium polygon for any convenient pole, starting on one of these lines and finishing on the other, a line through that pole parallel to the closing line of this equilibrium polygon will divide the resultant into two parts, equal to the components of the reactions parallel to the resultant, the other components being equal and opposite. If, then, the actual direction of one of the reactions be known, by drawing a line through this point of division of the resultant parallel to the line through the hinges, the actual reactions are determined.

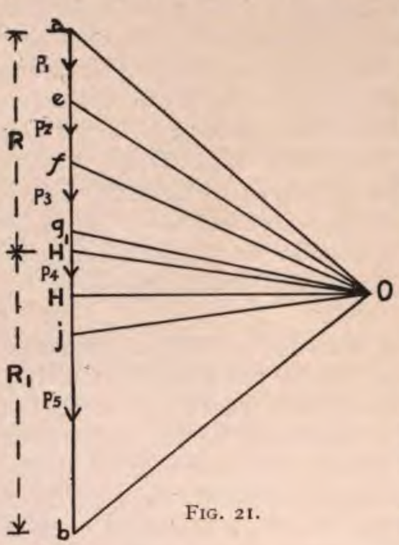


FIG. 21.

It will be seen that the force polygon closing is equivalent to the mathematical condition for equilibrium, that the sum of the resolved part of the acting forces in any direction is zero. Whilst the equilibrium polygon closing is equivalent to the mathematical condition, that the moment of the acting forces about any point equals zero ; for if this were not the case there would be a resultant couple which would be revealed by the first and last lines of the equilibrium polygon—*i.e.* the line from A parallel to H and the line from B parallel to H, being parallel and not coincident, and in this case, of course, there would not be equilibrium but rotation.

To avoid confusion it must be borne in mind that the components of the reactions spoken of are those parallel to the resultant and along the closing line, and not the total vertical and horizontal components unless the acting forces are vertical and the closing line is horizontal. When the resultant of the acting forces is vertical, the components of the reactions parallel to the resultant are not the total vertical reactions unless the

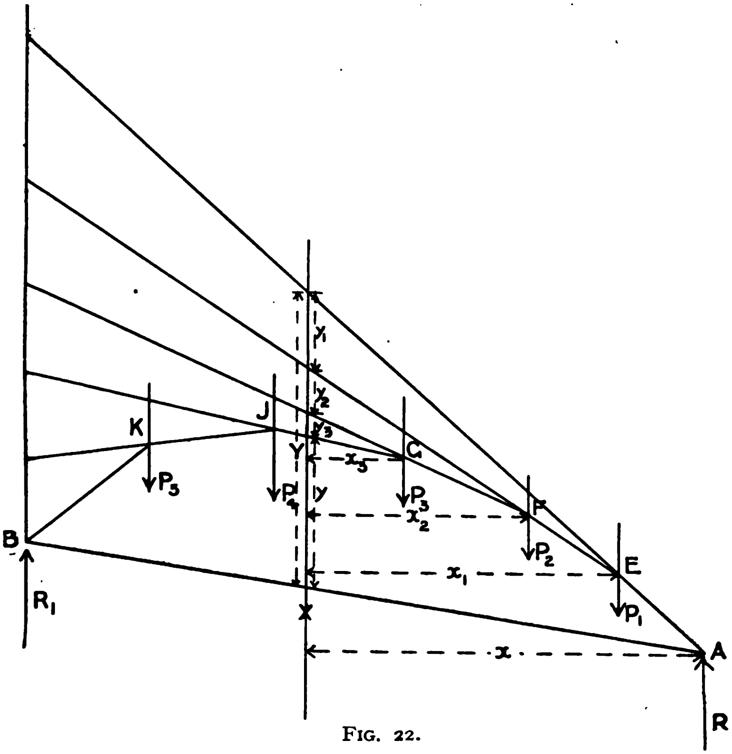


FIG. 22.

closing line is horizontal, because in other cases the component along the latter has a vertical component which will reduce the vertical component parallel to the resultant at the higher point of support, and increase it at the lower point of support. This is an important distinction, as will be seen in the case of arches fixed at the ends, or in the case of the supports being hinged but at different levels. When the acting forces are vertical and the closing line horizontal, as would be the case for an arch with hinged supports on the level, th- ents

of the reactions parallel to the resultant would of course be the same thing as the total vertical components of the reactions.

If the structure be subjected to vertical loads only, and hinged at the ends, or supported in an equivalent manner—*e.g.* like the cables of a suspension bridge supported on saddles and continued as back stays to the anchorages, let A and B (Fig. 22) be the positions of the supports, which need not be at the same level. To draw a true equilibrium polygon, the pole must be taken on a line parallel to A B through the point dividing the resultant into two parts equal to the resolved parts of the reactions along it, their other components being equal and opposite.

Now the components of the reactions parallel to the resultant of the acting forces, which in this case is vertical, will of course, as in the general case, be fixed by the position of the resultant relatively to the hinges, because their other components along the line of the hinges are equal and opposite, since the structure is in equilibrium under the action of the resultant of the acting forces and the two reactions. It in no way depends upon the position of the pole chosen. Therefore, suppose any pole O^1 to be taken and a line parallel to the resultant force $a\ b$, Fig. 21, drawn through B. Let a provisional equilibrium polygon for this pole O^1 (not shown in the figure) be completed, and take for the closing line the line joining A to the intersection of this polygon with the line through B parallel to the resultant force, and from O^1 draw a line parallel to it. This point will divide the resultant into two segments, which are the components of the reactions parallel to the resultant, which is vertical, the other components being equal and opposite along the closing line, and must therefore be the same point H^1, whatever pole is taken. From H^1 draw a line parallel to A B, and the true pole must lie on this line. If the direction of one actual reaction be known, from the corresponding end of $a\ b$ draw a line parallel to it to intersect the line from H^1 parallel to A B. This point is the true pole, O, and the lines joining it to a and b represent the true reactions in magnitude and direction.

As before, let the structure be subjected to vertical loads only and be hinged at the ends, the line joining the supports need not be horizontal:—

V.—*To prove, in this case, that the ordinate of the equilibrium polygon at any vertical section is proportional to the bending moment of the vertical forces there.* The vertical components

of the reactions in this case are not the total vertical components, but the two parts into which the closing line divides the resultant. Let $a\,b$ (Fig. 21) be the load line and from a plot down the loads consecutively, proceeding from A to B, let $b\,H^1$ be the component of the reaction at B parallel to the resultant and $H^1\,a$ that at A, the other components being along the closing line O H^1. These may be found either by drawing a provisional equilibrium polygon and drawing a ray from the pole parallel to its closing line, as just explained, or by taking moments about one support.

From H^1 draw a line parallel to A B, and take any point O on this line for the pole. Join O to a, e, f, g, j, b. Draw a vertical line at B. Starting at A, draw a line A E parallel to O a, from E draw a line E F parallel to O e, draw F G parallel to O f, and so on, till K B is parallel to O b. Let the lines A E, E F, F G, etc., be produced to meet the vertical line at B. Take a vertical section through any point X at a horizontal distance x from A, and let the intercepts on this line by the lines already drawn be denoted by y with suffixes corresponding to the forces as indicated in Fig. 22, the ordinate at X to A E produced being Y. Then we have from similar triangles, in Figs. 21 and 22, $\frac{Y}{x} = \frac{R}{H}$, R being the component of the reaction at A parallel to the vertical resultant—*i.e.* $a\,H^1$, and H the horizontal component of the other component H^1 of R along the closing line. Also—

$$\frac{y_1}{x_1} = \frac{P_1}{H},\ \frac{y_2}{x_2} = \frac{P_2}{H} \text{ and } \frac{y_3}{x_3} = \frac{P_3}{H},$$

where x_1, x_2, etc., are measured horizontally.

$$\therefore\quad Y = \frac{R \times x}{H}\ y_1 = \frac{P_1 \times x_1}{H}\ y_2 = \frac{P_2 \times x_2}{H} \text{ and } y_3 = \frac{P_3 \times x_3}{H}$$

If y is the ordinate of the equilibrium polygon at the section at X—

$$y = Y - y_1 - y_2 - y_3 = \frac{1}{H}\{R \times x - P_1 \times x_1 - P_2 \times x_2 - P_3 \times x_3)$$

$$= \frac{M}{H}, \text{ where M is the bending moment at the section X.}$$

Again, the ordinate of the equilibrium polygon multiplied by H is obviously the moment of the component, H^1, of the reactions along the closing line, because H^1 multiplied by the perpendicular on it from a point in the equilibrium polygon = H multiplied by the vertical ordinate of the point. So that the fact, that the ordinate multiplied by H = the bending moment due

to the vertical forces, is obvious if it be remembered that the tangent to the equilibrium polygon at any vertical is the line of the resultant force across that vertical; therefore, the algebraic sum of the moments of the force along the closing line and of the vertical forces on one side of the section, about the point where the vertical cuts the equilibrium polygon, must be zero; or, in other words, these moments are equal.

If A B is horizontal of course $H^1 = H$.

Thus it is clear that any ordinate of the equilibrium polygon multiplied by H = the bending moment of the vertical forces at that section. If, therefore, the ordinate y be measured in inches and multiplied by the number of feet represented by 1 inch = n say, in the scale of the span, and multiplied by H in tons, the result will be the bending moment in tons-feet, or $y \times 1'' \times n \times H$ = bending moment in tons-feet. Or the scale of bending moment is 1 inch = $n \times H$ tons-feet. Since the bending moment divided by the depth gives the horizontal component of flange stress in a girder, the scale to obtain this direct is 1 inch = $\frac{n \times H}{D}$ tons, where D is the depth of the girder at the point in feet.

It follows from the above that if an equilibrium polygon be drawn for any system of vertical forces on a girder supported at the ends and their reactions, the pole being taken on the line parallel to the hinges drawn through the point of meeting of the reactions in the force diagram, that :—

VI.—Any ordinate of this equilibrium polygon multiplied by H, the distance of the pole from the load line measured on the scale of the force diagram, equals the bending moment at that section.

VII.—To prove that if any side of the equilibrium polygon be produced to meet the vertical at one point of support, the ordinate intercepted on this vertical line $\times \frac{H}{L}$ = the component of the reaction at the other end parallel to the vertical resultant (when the other component is along the closing line), due to the load at the panel point where the line produced leaves the polygon, and to all the intermediate loads up to the vertical at the end. If the point X at which the section was supposed to be taken in the last paragraph were in particular at B, then $x = L$, and $y_1 = \frac{\text{the moment of } P_1 \text{ about } B}{H}$; similarly

$y_2 = \dfrac{\text{the moment of P}_2 \text{ about B}}{\text{H}}$, and so on for all the loads.
Now the moments of all the loads P_1, P_2, P_3, P_4, P_5 about B = the moment of the reaction at A about B.

Then $y_1 + y_2 + y_3 + y_4 + y_5 =$ the intercept on the vertical

$$\text{at B} = \frac{\text{R} \times \text{L}}{\text{H}};$$

therefore the intercept of A E produced on the vertical at B multiplied by $\frac{\text{H}}{\text{L}}$ equals the reaction at A due to all the loads from E to B. Similarly, if E F be produced, the intercept at B equals the reaction at A due to the loads at F, G, J, K, and so on.

Now it will be proved, page 51, that in a girder the maximum stress in the braces occurs when the longer segment of the girder up to the brace only is loaded, when dealing with loads at panel points; or if a typical load is considered, when the front of the load is some distance into the bay in question.

It also follows that at any vertical section the intercept at X, say, between the line A E produced and the equilibrium polygon, is proportional to the moment of all the acting forces from E to that section, because the intercept $= y_1 + y_2 + y_3$ $= \frac{1}{\text{H}} (\text{P}_1 x_1 + \text{P}_2 x_2 + \text{P}_3 x_3)$.

Thus, if the bridge be supposed to move relatively to the load, so that the left end of the bridge is now at X, the right end being a distance, equal to that of X from B, to the right of A, the only loads left on the bridge would be P_1, P_2, and P_3.

Therefore the reaction at the right extremity multiplied by $\text{L} = \text{P}_1 x_1 + \text{P}_2 x_2 + \text{P}_3 x_3$. In other words, the reaction at the right extremity equals the intercept between A E produced and the equilibrium polygon, multiplied by H and divided by L.

If A B be divided into panel lengths, the girder may be imagined to move relatively to the load instead of the load relatively to the bridge. If, for instance, there are seven panels, length b, and the leading load is supposed to be at the first panel point from A, we have found the reaction at A due to that position of the load, and that reaction would be the maximum shear in the end bay.

If, now, the head of the load moves back to the second panel point, measure $5\,b$ from E to the left, and measure the ordinate from A E produced down to the equilibrium polygon at this section, that intercept multiplied by H equals the moment of

all the loads between the second panel point and the section which represents the left end of the bridge when the head of the load is at the second panel point from the right. This intercept $\times \frac{H}{L}$ therefore equals the reaction at the right support under these conditions, and this would be the maximum shear in the second bay.

When the head of the load is at the third panel point from the right, measure $4\,b$ from E and the intercept on the ordinate at this point between A E produced and the equilibrium polygon $\times \frac{H}{L}$ = the reaction at the right support for this position of the load, which would be the maximum shear in the third bay. And similarly for any number of bay lengths.

If the scale of the span, Fig. 22, is n feet to an inch and the intercept measures y^1 inches, in order to obtain directly the shear, it is seen that this intercept represents $y^1 \times 1'' \times \frac{n \times H}{L}$, where H is in tons and L in feet, therefore the scale to obtain the shear in tons will be $\frac{n \times H}{L}$ tons to an inch.

To take an example, let the linear scale for the positions of the loads on the equilibrium polygon be 10 feet to the inch, and the scale used for the force diagram be 5 tons to the inch. If the object is simply to find the reactions, bending moments, or shearing forces, it has been shown that the position of the pole is only limited by the necessity of taking it on the line drawn, through the point of meeting of the reactions on the load line, parallel to the closing line; thus its distance from the load line can be taken any length convenient for simplifying the scales. If the closing line is horizontal and the distance of the pole from the load line measures 3 inches, then $H = 3 \times 5$ tons $= 15$ tons. Now if an ordinate of the equilibrium polygon measures 2 inches, the bending moment at that section equals $2 \times 1'' \times 10' \times \overset{T}{15} = 2 \times 1'' \times 150$ tons-feet—*i.e.* the scale of bending moments would be 1 inch = 150 tons-feet.

Since the bending moment divided by the depth gives the horizontal component of stress in the flanges when the section is taken at a point so as not to intersect a diagonal, in order to measure the horizontal component of flange stress directly from the equilibrium polygon—supposing the depth to be

7·5 feet, the scale is found as follows:—The flange stress at the section $= \dfrac{2 \times 1'' \times 10' \times \overset{T}{15}}{7{\cdot}5'} = 2 \times 1'' \times 20$ tons—*i.e.* the scale is 20 tons to the inch.

To scale off shearing force from the equilibrium polygon, if the span of the girder is 75 feet, suppose the intercept between the side produced and the equilibrium polygon at the section where it is measured is 7 inches. Then the shearing force to the right of the first load when the section taken represents the left end of the girder $= \dfrac{7 \times 1'' \times 10' \times \overset{T}{15}}{75'} = 7 \times 1'' \times$ 2 tons—*i.e.* the scale is 1 inch = 2 tons.

CHAPTER III

BRACED GIRDERS—PARALLEL TYPE

As the calculations for braced girders are simpler than for girders with continuous webs they will be considered first.

In the case of a braced girder the bending moment at a vertical section is balanced by the moment, about any centre, of the stresses in the members cut by the section. If the section be taken in Fig. 23, say, through an inclined brace and two flange members, the intersection of the inclined brace with one flange is the centre of moments to find the stress in the other flange, because the stresses in the brace and other flange would then not come in. The equation obtained by taking moments about this point is :—The stress, H, in the opposite flange multiplied by the perpendicular distance from the centre of moments (the depth, D, of the girder in the case of a parallel girder) must equal the bending moment, M, at the section. From which the stress in the flange is at once found. Again, if the vertical section be taken so as to intersect two flange members and one brace, the centre of moments in order to obtain the stress in the brace is obviously the point of intersection of the two flange members cut by the section. Now in a parallel girder this point would be at infinity in a direction parallel to the flanges, but this does not prevent it being used, as we are only concerned with the ratio of the length of the perpendicular P from it on to the resultant force, F, at a distance C to the right of the section (which perpendicular would be horizontal), to that of the length of the perpendicular on the line of the brace produced which equals $(P - C) \sin \theta$, when θ is the inclination of the brace to the horizontal. Now F is the shearing force at the section. Thus the stress in the brace

$$= \frac{\text{the shearing force at the section} \times P}{(P - C) \sin \theta}$$

$$= \frac{F \operatorname{cosec} \theta}{1 - \frac{C}{P}} = F \operatorname{cosec} \theta,$$

because $\frac{C}{P} = 0$.

Taking moments about the centre of moments, as above, is clearly equivalent to taking the section through the intersection of a brace with a flange and so as not to cut another brace, and applying the formula $M = H \times D$; and taking the section so as to cut a brace and two flanges and taking moments about the centre of moments at infinity, is equivalent to saying that the vertical component of stress in the braces must equal the shearing force.

If the stress in the upper flange is to be a pure compression and that in the lower flange a pure tension, the loads must only come on the main girders at the panel points—*i.e.* at the points where the braces are connected to the flanges.

To effect this result the load must be carried primarily on cross-girders which are supported on the main girders at the panel points. If the load comes on to the flange direct, as it would do in the case of trough flooring being used instead of cross-girders, that member of the flange will act as a short girder between the panel points in addition to forming a tension or compression member, as the case may be, of the bridge,—unless the web is continuous, in which case the girder becomes a plate girder. Since the vertical component of stress in the bracing members must everywhere balance the shearing force, those members must be connected end to end without break from one extremity of the main girder to the other, unless the shearing force becomes and always remains zero for any length. Thus the braces transmit the load on the girder to the two points of support, and the flanges serve to hold the braces in position.

In what follows it is supposed that the load is transmitted to the main girders by cross-girders at the panel points, when the shearing force between the panel points will be constant, as no external load comes on to the main girders between two consecutive panel points. It will be noticed that the formula $M = H \times D$ is applicable whether the flanges of the girder are parallel or not; H always being the horizontal component of the total stress in the flange in a vertical section which does not cut an inclined member of the bracing, D being the distance from centre of gravity to centre of gravity of the flanges at that section.

PARALLEL GIRDERS

If D *is constant* then $H \propto M$, or since $H = f \times B \times T$, where f is the intensity of horizontal stress in the flange, B is the width of the flange and T its thickness, we have $f \times B \times T \propto M$. If the breadth of the flange and the working intensity of stress

are kept constant $T \propto M$, or, in words, the thickness of the flange varies as the bending moment. If H *is constant*—*i.e.* the horizontal component of stress in the flanges is constant—then $D \propto M$, or the depth varies as the bending moment. This case will be considered later.

Stresses due to dead load

It must be pointed out, in the first place, that the stresses have to be calculated for two different kinds of loads, "dead" load, which is always constant, and "live" load, which varies and is sometimes altogether absent—it is generally a load that travels over the bridge. To find the stresses due to dead load, first find the weight of the main girders and the platform load (the latter consisting of the cross-girders, rail-bearers, or stringers—if used, the flooring, rails, and sleepers, etc.), and the wind bracing. A very approximate idea of the weight can generally be obtained from that of existing bridges of about the same span, but if the weight of the bridge when designed is found to be materially different, the stresses will have to be corrected accordingly. The platform load and wind bracing are still more readily obtained in the same way. The weight of the bridge can generally be taken as a uniform load. the panel load due to the weight of the main girders is taken as acting half at the upper and half at the lower panel points; the platform load is taken as acting at the upper or lower panel points, according as the load is carried on the upper or lower flange, and the weight of the wind bracing should be taken as acting at the panel points of the other flange, as separate wind bracing will not generally be required for the flange which carries the platform.

If, therefore, W be the panel load due to the weight of the two main girders—*i.e.* the weight of the two main girders divided by the number of panels in each girder—W' that due to the total platform load, and W'' the panel load due to the wind bracing; if the load is carried on the lower flange, calling the load at each upper panel point of the girder P and that at each lower panel point P'—

$$P = \frac{W}{4} + \frac{W''}{2},$$

—and

$$P' = \frac{W}{4} + \frac{W'}{2}.$$

If the load is on the upper flange, these will be reversed.

From a consideration of the equilibrium of any panel point it is obvious, since the point is at rest, that the difference of the vertical components of the stress in the braces meeting at that point must balance the load carried at the joint. Also the difference between the stresses in the two panel lengths of flange meeting at the joint must be equal to the differences of the horizontal components of stress in the braces meeting there ; if one brace is vertical the horizontal component of that stress is zero.

With these premises the stress in the members in each bay may be calculated in either of three ways, exclusive of the graphic method.

I.—(*a*) By considering the equilibrium of each panel point in the vertical direction, the vertical component of stress in each member of the bracing may be found.

(*b*) Knowing the vertical component of stress in the inclined braces, their horizontal components of stress equal the vertical components multiplied by cot θ, θ being the angle the brace makes with the horizontal.

(*c*) Knowing the horizontal components of stress in the braces and starting at either end, the horizontal stress in the lengths of flange to which the brace is connected can be found by considering the equilibrium of the joint in the horizontal direction.

II.—(*a*) The bending moment at each joint may be found by taking the moments of all the forces acting to the right of the joint. Dividing these by the depth, the total horizontal stress at each end of the vertical section is found; on the side of the vertical where no brace is connected this would be the total stress in that length of flange.

(*b*) The difference of the stresses in two consecutive lengths of flange equals the difference of the horizontal components of stress in the braces connected to the flange at that point; if one of the braces is vertical it would equal the horizontal component of stress in the other.

(*c*) Knowing the horizontal components of stress in the braces, the vertical components are found by multiplying the former by tan θ.

(*d*) By considering the vertical equilibrium of each joint we obtain the stresses in the verticals.

III.—(*a*) By considering the shearing force in each panel or bay, which is constant throughout the bay, we know at once the vertical component of stress in the inclined brace in the bay; this enables the stress in the verticals to be found by considering the vertical equilibrium of the joint at either of its extremities.

(*b*) The horizontal component of the stress in the inclined members can be found by multiplying the vertical component by cot θ.

(*c*) Starting at the ends this enables the stresses in the flange members to be deduced by considering the horizontal equilibrium of the joints.

As the last method serves best to keep in mind that the members of the bracing serve to balance the shearing force, it will be made use of in what immediately follows.

If the loads acting at any vertical be considered, it will be seen that the bracing transmits the weight of each load to the abutments in the inverse proportion of the distances of the load from the abutments. If in Fig. 23 we consider the load P acting at the first panel point from the right-hand end of the girder, evidently $\frac{5}{6}$ P must be transmitted to the right abutment, and $\frac{1}{6}$ P to the left abutment. This $\frac{5}{6}$ P being transmitted to the lower end of the inclined brace to the right of it will cause a tension in that brace; but the $\frac{1}{6}$ P being transmitted to the upper extremity of the members to the left up to the centre will tend to cause a compression of the inclined braces to the left up to the centre, and, as the inclination of the braces is reversed at the centre, a tension in the braces beyond the centre. Now consider the load P acting at the first panel point from the left-hand end of the girder, evidently $\frac{5}{6}$ P must be transmitted to the left abutment, causing a tension in the brace to the left of it, and this will be added to the tension of $\frac{1}{6}$ P from the load first considered. One panel distant from the left abutment $\frac{1}{6}$ of the load is transmitted to the right abutment, causing compression in the braces to the right of it up to the centre, and, since the inclination of the braces is there reversed, tension in the inclined braces to the right of the centre, which, in the brace in the bay adjoining the right abutment, will be added to the $\frac{5}{6}$ P due to the load first considered. Thus the portions of the loads P, distant one panel length from the ends, transmitted towards the centre cause stress in the intermediate members which counteract each other, but in the end braces

the stresses due to the two portions of the loads, $\frac{5}{6}$ P and $\frac{1}{6}$ P, add together, and taking the two loads P, respectively one panel distant from the ends of the girder, together, the stress in the braces between each load and the ends are the same as if each load were transmitted to the nearest abutment. Generally,

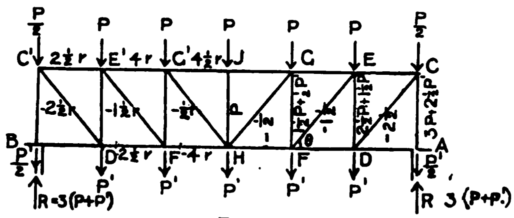


FIG. 23.

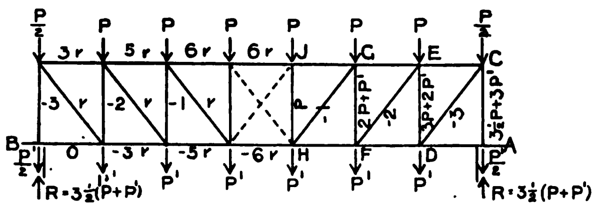


FIG. 24.

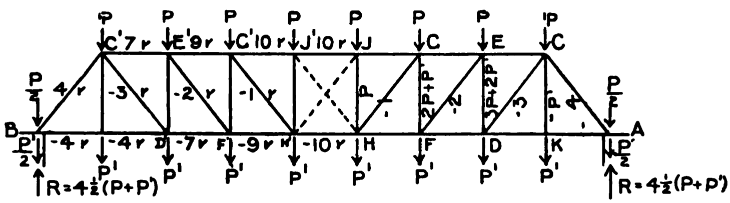


FIG. 25.

if two loads are symmetrically placed with respect to the ends of a girder the stresses in the braces between the loads and the ends will be the same as if each load were transmitted to the nearest abutment. Thus with equal loads at each panel point on the upper and lower flanges we may imagine each load to be transmitted to the nearest abutment; if there is a panel point at the centre the loads there would be transmitted half to each abutment. With a symmetrical load we may, therefore, start in the middle and find the vertical component of stress in each inclined brace by assuming that half the load at the centre and the loads between it and the centre are trans-

mitted through it, but as this method is not applicable to unsymmetrical loading, it is best to find the reactions at the ends and the shearing force in each bay, and then the method for an unsymmetrical load is exactly similar.

In Figs. 23 and 24, the N type of girder is depicted, the reactions are of course half the total load, and—
the vertical component of stress in the first inclined brace $=$ the shearing force in that bay $=$ the reaction $- \frac{1}{2}(P + P^1)$;
the vertical component of stress in the second inclined brace $=$ the shearing force in that bay $=$ the reaction $- \frac{3}{2}(P + P^1)$;
the vertical component of stress in the third inclined brace $=$ the shearing force in that bay $=$ the reaction $- \frac{5}{2}(P + P^1)$.
If the girder had more bays, the vertical components of stress in the remaining braces would be found in a similar manner.

The stress in the vertical equals the vertical components of the stress in the inclined brace joined to it at its top plus the load on the top of the vertical, as is seen by considering the equilibrium of the joint in the vertical direction.

The vertical components of stress in the inclined braces $\times \cot \theta$ (which will be denoted by r for brevity), represent the horizontal components of stress in these braces.

Taking the left side of the girder, the stress in the flange in any bay equals the stress in the flange in the bay to the left of it $+$ the horizontal component of stress in the inclined brace attached to its left extremity, as is seen by considering the equilibrium of that joint in the horizontal direction.

In Fig. 23 there are six bays or panels, therefore the reactions equal $\frac{6}{2}(P + P^1) = 3(P + P^1)$. As explained above, the shearing force in the first bay $= 3(P + P^1) - \frac{P + P^1}{2} = 2\frac{1}{2}(P + P^1)$, *i.e.* the reaction minus the loads acting at A and C, this is therefore the vertical component of the stress in C D, it is a tension as the shear is positive because it tends to lift the portion of the girder to the right of a section. The compression in A C is found as explained above, because it is equal to this vertical component $+$ the load acting at its upper extremity—*i.e.* it is $2\frac{1}{2}(P + P^1) + \frac{P}{2} = 3P + 2\frac{1}{2}P^1$. The shearing force in the second bay equals $3(P + P^1) - 1\frac{1}{2}(P + P^1) = 1\frac{1}{2}(P + P^1)$, which is the vertical component of stress in E F, and is a tension as in the case of C D, because the shearing force is positive. The compression in E D $=$ this vertical component $+ P$—*i.e.* $2\frac{1}{2}P + 1\frac{1}{2}P^1$. The

shearing force in the third bay $= 3\,(P + P^1) - 2\frac{1}{2}\,(P + P^1) = \frac{1}{2}\,(P + P^1)$, which is therefore the vertical component of stress in G H. The compression in G F equals this component + the load at the top, P,—*i.e.* $1\frac{1}{2}\,P + \frac{1}{2}\,P^1$. The compression in H J = P, as there is no inclined brace attached at its top. In Fig. 23 these vertical stresses are written on the right half of the diagram. The total stresses in the inclined braces equal the vertical components multiplied by cosec θ. The horizontal components of the stress in the inclined braces are obtained by multiplying their vertical component by r (cot θ); these results are written on the left half of the diagram, the stresses in the members of the two halves of the girder due to dead load are, of course, the same. The stresses in the flanges in the various bays can now be written down, by considering the horizontal equilibrium of the joints as explained above; in the figure, when the stress is a multiple of $P + P^1$, $P + P^1$ is omitted to avoid cramping the space available. The stress in $C^1\,E^1$ equals $2\frac{1}{2}\,(P + P^1)\,r$ the horizontal component of stress in $C^1\,D^1$. The stress in $E^1\,G^1$ = that in $C^1\,E^1$ + the horizontal component in $E^1\,F^1 = 4\,(P + P^1)\,r$. The stress in $G^1\,J$ = that in $E^1\,G^1$ + the horizontal component in $G^1\,H = 4\frac{1}{2}\,(P + P^1)\,r$. In the lower flange there is no stress in $B\,D^1$ due to the dead load, therefore in $D^1\,F^1$ the stress equals the horizontal component in $C^1\,D^1 = 2\frac{1}{2}\,(P + P^1)\,r$. In $F^1\,H$ the stress equals that in $D^1\,F^1$ + the horizontal component in $E^1\,F^1 = 4\,(P + P^1)\,r$. The stresses in the members of the upper flange are obviously compressions, and in the members of the lower flange tensions. It will be noticed that the stress in the members of the upper and lower flange lying between a pair of inclined braces are the same. It is obvious from the method of arriving at them that the stresses in the inclined braces and the flanges will be the same whether the loads are applied at the upper or lower panel points or at both, but the stresses in the verticals are greater when any part of the load is applied at their upper extremity, by the amount of that portion of the load. The stresses in the braces increase from the centre to the ends, and those in the flanges from the ends to the centre, at the point of attachment of each inclined brace.

Fig. 24 represents the same type of girder but with an odd number (7), instead of an even number, of bays, so that if the panel length is the same as before, the length of the girder is increased in the ratio of 7 to 6, and therefore the stresses in the flanges will be increased, due to this cause. The reactions

in this case are $\frac{7}{2}$ (P + P^1), and finding the shearing force in each bay as in the last case, the vertical components of the tensions in the inclined braces and the compressions in the verticals are directly found. These are written on the right half of the diagram, as in Fig. 23. On the left half, the horizontal components of the stress in the braces obtained by multiplying the vertical components by r (cot θ) are written, and these are added to obtain the flange stresses there noted. Fig. 25 is a slightly modified form of Figs. 23 and 24, known as the Pratt truss. It is obtained by adding a bay at each end of the previous type, but there is no vertical at the ends, and no member of the upper flange in the end bays, which effects some saving in weight of metal. The platform is, of course, carried on the lower flange. In Fig. 25 nine bays are shown, so that for the same panel length the girder is longer than that in Fig. 24 in the proportion of 9 : 7, and the flange stresses are increased accordingly. The vertical components of the stress in the inclined braces are found by writing down the shearing force in each bay as before, these are written on the right half of Fig. 25. A C is in compression, as it forms part of the upper flange; C K is in tension, due to the load P^1 hung from its lower extremity, the other inclined members are in tension, and the other verticals in compression. The resultant tensions in the former and the compressions in the latter are found exactly as before. On the left half of Fig. 25 the horizontal components of the stresses in the inclined members, obtained by multiplying the vertical components by cot θ, are written, and these are added to find the stress in the members of the flanges, or, in other words, the latter stresses are obtained by considering the horizontal equilibrium of the joints. The stress in C^1 E^1 is the sum of the horizontal components in B C^1 and C^1 D^1 = 7 (P + P^1) r. The stress in B D^1 equals the horizontal component in B C^1 = 4 (P + P^1) r. The remaining stresses are obtained as before.

A type of girder is very often made use of which consists of an N-type girder of an even number of bays superimposed on or coalesced into one of an odd number of bays, known as a Linville truss. The foot of the end verticals of the smaller number of bays is connected by an inclined brace (the co-tangent of whose inclination is necessarily one-half that of the other braces) to the top of the end verticals as shown in Fig. 26. The girder thus consists of an upper and lower flange, and a double system of bracing as indicated in Fig. 26. Each component

girder is assumed to take its own share of the load, so that the stresses in the bracings will be the same as for the girders taken separately; but P and P^1, it will be noticed, are now the loads on a panel of one-half the length of those in the previous cases. The stresses in the members of the flanges will, of course, be due to both systems of bracing. The compression in the verticals, except the end verticals, and the vertical and horizontal components of the tensions in the inclined braces, except the steeper braces at the end, may be written down from Figs. 23 and 24. The vertical component of stress in the steeper inclined braces at the ends equals that in the vertical connected to its foot $+ P^1 = 3\frac{1}{2}(P + P^1)$ (in this case a load P acts at the upper extremity of the vertical instead of $\frac{P}{2}$, as in Fig. 23); therefore the stress in the end vertical is greater than $\frac{7}{2}P + 3P^1$, its value for the girder in Fig. 24, by this amount, and therefore equals $7P + 6\frac{1}{2}P^1$. The horizontal component of stress in the steeper braces equals the vertical component $\times \frac{r}{2}$, because the co-tangent of its inclination is half that for the other braces—*i.e.* $3\frac{1}{2}(P + P^1)\frac{r}{2} = \frac{7}{4}(P + P^1)r$. The horizontal component in the other inclined brace connected to the end post $= 3(P + P^1)r$, therefore the top flange stress in the first bay, $4\frac{3}{4}(P + P^1)r$, and the stress in the other bays is found by adding the horizontal components of the stresses in the inclined braces where they are joined to the flanges.

It will be noticed that if the span is the same in this case as in Fig. 24, the stress figures on the diagram in Fig. 26 should be divided by 2 to compare with those on the diagram in Fig. 24. Also that the average stress in the flange lengths of Fig. 26 in any double bay are equal to the flange stress in the corresponding bay in Fig. 24, but the stresses in individual members of the bracing in Fig. 26 are one-half those of the corresponding braces in Figs. 23 and 24. The design in Fig. 26 is therefore advantageous when the size of the web members in the former design would be greater than could be conveniently worked in, the disadvantage is that the double system of bracings renders the calculated value of the stresses less reliable. The last type of girder is often modified by introducing an additional bay at each end, much as in the Pratt girder, but of the shorter bay length as

shown in Fig. 27. This design is known as the Whipple truss, and eliminates the end verticals and top flange members in the end bays. The vertical component of stress in the end inclined members $= (3 + 3\frac{1}{2} + 1)\ (P + P^1) = 7\frac{1}{2}\ (P + P^1)$, therefore the horizontal component in it equals $7\frac{1}{2}\,(P + P^1)\,\frac{r}{2} = 3\frac{3}{4}\,(P + P^1)\,r$. The stresses in the other inclined braces and in the verticals (except the verticals nearest to the ends which are

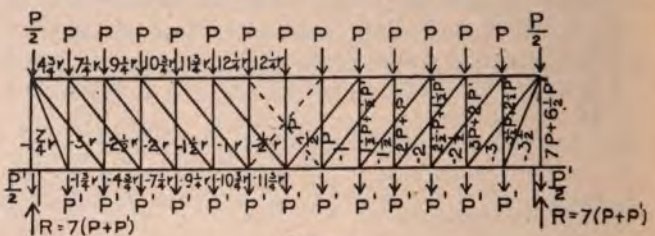


FIG. 26.

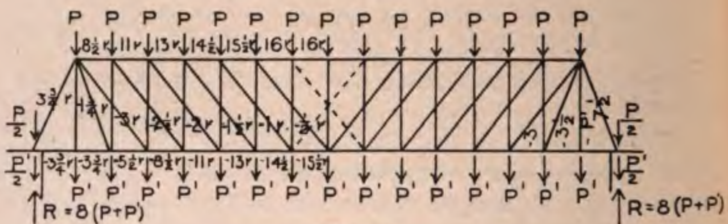


FIG. 27.

in tension, due to the load P^1 at their lower extremity) are the same as in Fig. 26. The horizontal components of the stresses in the inclined members are written on the left half of Fig. 27. In the first upper flange member the stress is $(3\frac{3}{4} + 1\frac{3}{4} + 3)$ $(P + P^1) = 8\frac{1}{2}\,(P + P^1)$; and in the first two bays of the lower flange the stress is $3\frac{3}{4}\,(P + P^1)$. The stresses in the remaining flange lengths are obtained by adding consecutively the horizontal components of the stresses in the braces at their points of attachment. The increased value of the stress in the flanges compared with the last case is simply due to the increase of span, which is greater than in Fig. 26, with the same bay length, in the proportion of 8 to 7. The designs indicated in Figs. 26 and 27 have the advantage of shortening the stringers between the cross-girders, but the double system of bracing is a disadvantage in that the stresses are not determined with the same certainty

a single system, seeing that their values depend upon the cy of the assumption that each of the systems take up ‹e share of the load. The same advantage of shortening ringers without the disadvantage of introducing indefinite-

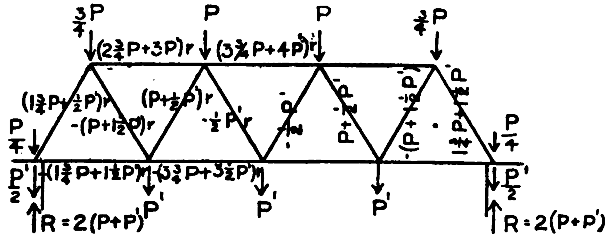


FIG. 28.

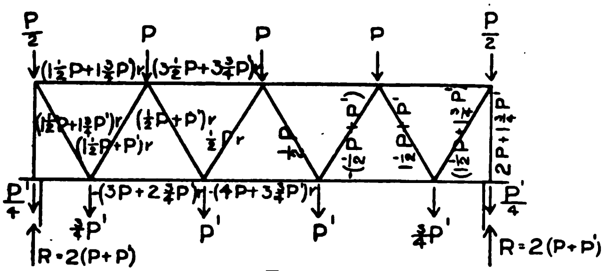


FIG. 29.

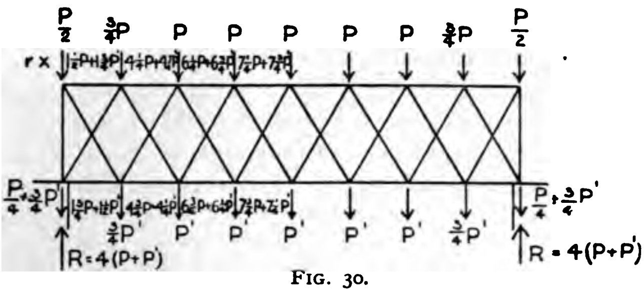


FIG. 30.

ness in the amount of stress taken up by the web members is obtained in what is known as the Baltimore truss, in which sub-verticals are introduced, hung from the centre of the inclined braces, this point being tied to the top of the next post or vertical.

It will have been noticed that in the types of girder considered above the vertical members of the web bracing are in compression, and the inclined members in tension—*i.e.*

struts are of the shortest length possible. If, however, the braces are equally inclined to the horizontal, alternately in opposite directions, as in the Warren girder, this advantage no longer obtains, but the type nevertheless is of considerable importance, and it will therefore be shortly considered. Figs. 28 and 29 indicate modifications of this form of girder for carrying the load on the lower flange. First find the shearing force between the line of action of each vertical load; this will represent the vertical component of the stress in the corresponding brace, and is written on the right-hand half of the figures. From these the horizontal components are found by multiplying by r (cot θ), and are written on the left-hand half of the figures. By adding the horizontal components of the stresses in the braces at the points where they are attached to the flanges, the stresses in the latter are determined, and are written on the figures. In Fig. 28 there is an odd half bay at each end of the upper flange, and in Fig. 29 at each end of the lower flange; hence in the former case for the upper flange and in the latter case for the lower flange, the load at the first panel point from the end is one-half the panel load from one side and one-quarter the panel load from the other side, or three-quarters the panel load altogether. It will be noticed that the stresses in the two figures are identical when one is inverted and P and P^1 are interchanged.

These two outlines are sometimes combined to form a single girder (Fig. 30) when it is assumed that each set of bracing takes half the load, and it must be borne in mind that if the span and loading is the same as before, P and P^1 will have half their original values, so that to compare the stresses in Fig. 30 with those in Figs. 28 and 29, the former must be divided by 2. The horizontal component of the stresses in the braces may be written on the diagram from Figs. 28 and 29, and the flange stresses written down by adding these horizontal components at their points of connection to the flanges.

The effect of using the last form of girder is to reduce the effective bay length to one-half, but the double system of bracing introduces the indefiniteness above alluded to as to the actual stress in any brace. The same advantage may be obtained without this defect by introducing verticals, if the load is carried on the bottom flange—at the top vertices of the triangles, and if the load is carried on the top flange—at the bottom vertices of the triangles, as indicated in Figs. 31 and 32.

Writing down the shearing forces between the lines of action of the loads as before, we obtain the vertical components of stress in the diagonals, from these their horizontal components, and thence the flange stresses are arrived at, as indicated in Figs. 31 and 32.

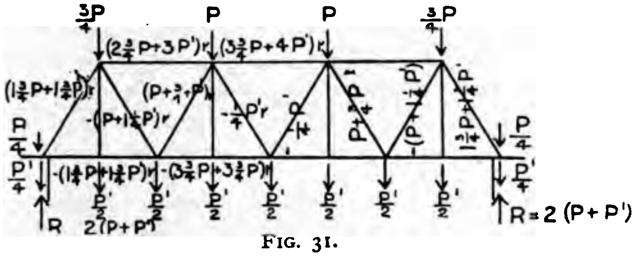


FIG. 31.

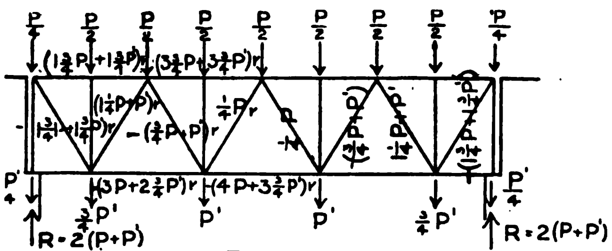


FIG. 32.

There are several forms of truss which have been used for bridges, some of which are still used in the case of roofs and for temporary purposes, but which involve for bridge work the use of comparatively long tie rods, leading to lack of rigidity.

BOLLMAN TRUSS

First may be considered the truss consisting of vertical struts fixed at right angles to the horizontal member, the other

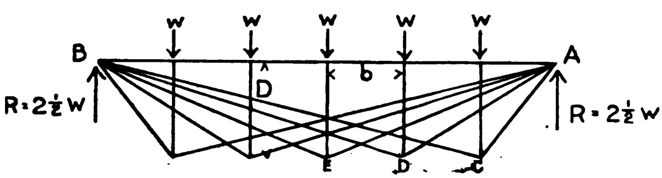


FIG. 33.

ends of the struts being tied by tension rods to the ends of the horizontal member, known as the Bollman truss.

If we suppose the horizontal member to be divided into six equal parts by five vertical struts of equal length, the free ends of which are tied to the ends of the horizontal member, it will be seen that the truss (Fig. 33) consists, in effect, of five triangles, the horizontal member A B being common to all, and therefore subject to a stress from each of the triangles considered separately. Let b be the panel length, D the depth of the verticals, and W the panel load. Consider first the triangle A B C. The vertical component of stress in A C is obviously the reaction at A, due to the load acting in the vertical through C—*i.e.* $\frac{5}{6}$ W—and the vertical component in B C is $\frac{1}{6}$ W. The horizontal component of stress in either is therefore $\frac{5}{6}\frac{b}{D}$ W, which is also necessarily the stress in A B, due to the triangle A B C. Next consider the triangle A D B. The vertical component of stress in A D is clearly $\frac{2}{3}$ W and in B D it is $\frac{1}{3}$ W, consequently the horizontal component of stress in either, and therefore in A B, due to the triangle A B D, is $\frac{2}{3} W \times \frac{2b}{D} = \frac{4}{3}\frac{b}{D}$ W. Due to the triangle A E B the horizontal stress in

$$AB = \frac{W}{2} \times \frac{3b}{D} = \frac{3}{2}\frac{b}{D} W.$$

$\therefore$ the total stress in A B $= \frac{b}{6D} W (5 + 8 + 9 + 8 + 5)$

$= \frac{35}{6}\frac{b}{D} W.$

Similarly, if the horizontal member is divided into eight parts the total stress in A B $= \frac{b}{8D} W (7 + 12 + 15 + 16 + 15 + 12 \times 7)$ $= \frac{84}{8}\frac{b}{D} W.$ The total stress in the tension rods equals the vertical component of stress in them multiplied by the cosec of their inclination to the horizontal, and the compression in the vertical struts equals W.

TRAPEZOIDAL TRUSS

Instead of supporting points on the horizontal member by struts held up by triangular trusses, trapezoidal trusses may be used, and the truss represented in Fig. 34, known as the trapezoidal truss, results. Each trapezoid supports a pair of struts at equal distances from the centre. If the horizontal member is divided into an equal number of parts, the middle

truss will be a triangle, and the vertical component of stress in A C will be $\frac{W}{2}$, and the horizontal component of stress is $\frac{W}{2}\frac{3\,b}{D} = \frac{3}{2}\frac{b}{D}W$.

Consider next the trapezoid A B D^1 D, the vertical component of stress in A D = W, and the horizontal component in it $= W \times \frac{2\,b}{D} = 2\frac{b}{D}W$, which is therefore the stress in A B and D D^1 due to this trapezoid. Taking next the trapezoid A B E^1 E,

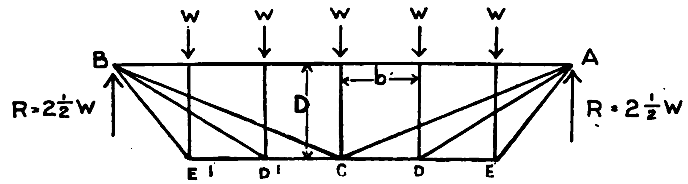


FIG. 34.

the vertical component of stress in A E = W, and the horizontal component of stress in it $= W \times \frac{b}{D}$; this, therefore, is the stress in A B and in E E^1 due to this trapezoid. The total stress in A B is therefore $= \frac{b}{D}W(1\tfrac{1}{2} + 2 + 1) = 4\tfrac{1}{2}\frac{b}{D}W$, which is less than the value in the previous case. The stress in D D^1 = $\frac{b}{D}W(2+1)$, and that in E D and E^1 D^1 $= \frac{b}{D}W$.

For eight bays the stress in A B $= \frac{b}{D}W(2 + 3 + 2 + 1) = 8\frac{b}{D}W$.

FINK TRUSS

The horizontal member may be supported at the centre by a triangular truss, as in the first case, but instead of supporting other points by rods extending to the ends, the centre of each half span may be supported by a triangular truss, and again the middle of each quarter, and so on, as in Fig. 35.

In the smallest trusses one-half W is transmitted through each of the shortest inclined rods, the vertical component of stress in which is therefore $\frac{W}{2}$. In the trusses containing two

lengths of the smallest trusses, the two latter each transmit a load $\frac{w}{2}$ to its centre; therefore the compression in its centre vertical equals $w + 2\frac{w}{2} = 2\,w$, and therefore the vertical component of stress in each of the intermediate length rods $= w$. The

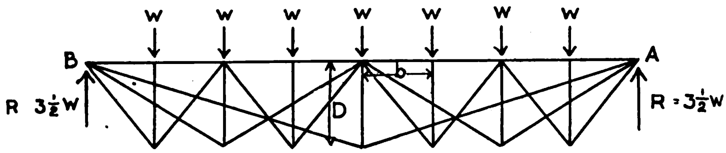


FIG. 35.

longest truss contains two lengths of the intermediate truss, which each transmit a load w to its centre; also the two small trusses on either side of its central strut each transmit $\frac{w}{2}$ to its centre; therefore the vertical compression in its central rod $= 2\,w + 2\frac{w}{2} + w = 4\,w$, therefore the vertical component of stress in each of the longest ties equals 2 w. The horizontal component of stress in the shortest ties is therefore $\frac{b}{D}\frac{w}{2}$, in the medium ties $\frac{2\,b}{D}\,w$, and in the longest ties $\frac{4\,b}{D} \times 2\,w = 8\frac{b}{D}\,w$. The total stress in A B is due to the sum of these three horizontal components, as each part of A B forms a portion of one of each of the three sizes of trusses; therefore the total stress in A B $= \frac{b}{D}\,w\,(\frac{1}{2} + 2 + 8) = 10\frac{1}{2}\,\frac{b}{D}\,w$, the same value as in the case of the Bollman truss.

It will be noticed in all the last three types of truss that some of the braces are of great length, and consequently small inclination; this causes the resultant stress in such members to be considerable, which makes these types uneconomical.

Stresses due to live load

Since the dead load is constant, the stresses in all the members of the girder due to it must also be constant, but the stresses in the different members due to the live load may, and actually do, in the case of the braces, become a maximum for different

positions of the live load relatively to the bridge. With respect to the members of the flanges, since the horizontal component of stress in them equals $\frac{M}{D}$, where M is the bending moment at the intersection of the inclined brace with the other flange, it is obvious that the flange stress will be a maximum when M is a maximum at the section in question, because if the live load be considered as a series of individual loads on the bridge, the bending moment diagram for each load, taken separately, would be a triangle with its vertex in the vertical line of the load, and if for a given number of loads these triangles be all plotted on the same base, representing the span of the bridge, the total bending moment at any section due to these loads equals the sum of the ordinates of the triangles at that section, consequently, if the load consists of a series of equal loads, or if it is uniform, the more loads there are on the bridge the greater will be the bending moment at any point;—*i.e.* for such a series of loads the bending moment at every panel point, and therefore the stress in the flange, is greatest when the load covers the bridge. If the load consists of unequal concentrated loads, the position of the load giving the maximum bending moment in each bay has to be determined.

But the case is different for members of the bracing, for if the effect of loads on the longer segment, say to the right of the inclined brace under consideration in an N girder bridge, be examined into, it will be evident that the portion of such loads on that segment

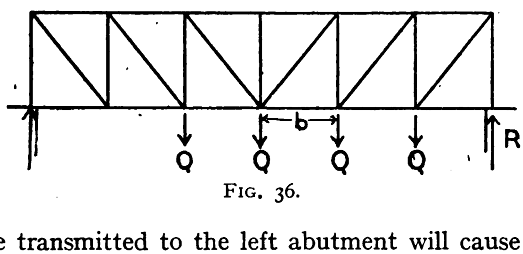


FIG. 36.

which are transmitted to the left abutment will cause a tension in the members in question. Supposing, for instance, we consider the second brace from the left end in Fig. 36, $\frac{4}{6}$ of the first load to its right, $\frac{3}{6}$ of the second load, $\frac{2}{6}$ of the third load, and $\frac{1}{6}$ of the fourth load are transmitted to the left, and cause a tension in the second and first brace from that end whose vertical component equals $\frac{10}{6}$ W. If there were a load on the first joint, $\frac{1}{6}$ of this load would be transmitted to the right, and that would

produce a compressive stress in the second inclined brace and reduce the vertical component of the tension in it from $\frac{10}{6}$ w to $\frac{9}{6}$ w. Thus it will be seen that the maximum tension occurs when the loads are on the longer segment only. It will be observed that a load w has been taken at each of the panel points at which there is a load, but if the load were a uniformly distributed one, there would only be $\frac{w}{2}$ at the panel point at the head of the load, and the reaction at the left abutment, which is equal to the shear in the first two bays in the example considered, would equal $\frac{w\,(4\,b)^2}{2 \times 6\,b} = \frac{8}{6}\,w\,b$, instead of $\frac{9}{6}$ of the panel load as before determined for a load w at each joint. But if the load is carried on cross-girders which are themselves supported on the main girders, there would be a greater tension in the brace in question when the uniform load has advanced a certain distance beyond the panel point.

In Fig. 37 suppose that a uniform load rests on the stringers or rail-bearers which are supported on the cross-girders between

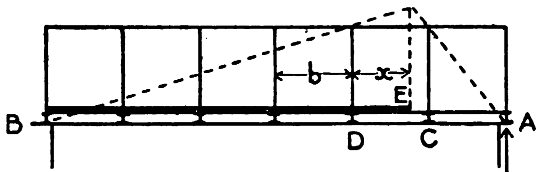


FIG. 37.

C and B. If a point E be found at which a concentrated load will increase the reaction at A in the same proportion that it will increase the load brought on the main girder by the cross-girder at C (which is equal to the reaction of the stringers C D on the cross-girder at C, due to the concentrated load at E), the shear on the main girder in the bay C D would not be affected by such a load. This is obvious, because the shearing force in that bay equals the total reaction at A minus the load transmitted to the main girder at C. But if a concentrated load be placed anywhere between E and D, the reaction of the stringer at C will be increased in a less ratio than the reaction at A, therefore it will increase the shearing force in the bay C D, and by similar reasoning a load between E and C would decrease the shearing force in the bay C D because the reaction of the stringer at C would be increased more than the reaction at A by such a load.

This being the case, the shear at D will be a maximum if the load extends to E, where $\frac{DE}{CD} = \frac{BE}{AB}$, let $x = DE$, then $\frac{x}{b} = \frac{4b + x}{6b}$ $\therefore$ $6bx = 4b^2 + bx$, or $x = \frac{4}{5}b$.

The position of E can obviously be found by joining A and B to the upper extremities of the verticals at C and D respectively, producing these lines till they meet and dropping a vertical from the point of intersection.

The shearing force at D equals the reaction at A minus the load brought on the main girder by the cross-girder at C.

$$\therefore \text{ the shearing force at D} = \frac{w\,(4b + \frac{4}{5}b)^2}{2 \times 6b} - \frac{w\,16\,b^2}{25 \times 2 \times b}$$
$$= \frac{48}{25}wb - \frac{8}{25}wb = \frac{8}{5}wb,$$

where $wb = W$, the panel load.

The difference between the value of $\frac{8}{5}W$ and the value $\frac{10}{6}W$ found by considering a load W at the panel point $D = \left(\frac{10}{6} - \frac{8}{5}\right)W = \frac{W}{15}$. With a greater number of bays in the girder, the difference would be still smaller in the end bays.

It is thus seen that by taking a load W at each panel point, instead of an equivalent uniform load, the calculated values of the shear will be a little too high, and therefore on the safe side, and the excess will be greatest in the centre bays where the proportionate range of the stress is the greatest.

There are two methods of designing a girder, subject to live load.

(1) For a uniform load which is equivalent to the maximum actual load—*i.e.* a uniform load which gives a bending moment at each point of the span, which is at least as great as that given by the maximum load the bridge is likely to have to carry. To find the "equivalent" uniform load, the bending moment diagrams for the different positions of the actual loads on the bridge have to be found and plotted, and a parabola drawn which falls everywhere outside these diagrams. This parabola is the bending moment diagram of the "equivalent" uniform load.

(2) For a typical load which is chosen as causing stresses at least as great as the maximum stresses due to any actual load likely to come on to the bridge.

(1) Taking the former method first—*i.e.* to design a bridge for an "equivalent" uniform load. It will be obvious that the equivalent uniform load per foot will be smaller the longer the span. In calculating it two locomotives coupled together, followed by a train of waggons with their maximum loads, are generally considered. The load per foot due to the waggons is less than that due to the locomotives, consequently, in a long bridge, a considerable proportion of its length would only be subjected to the less intense load of the waggons; but as the span decreases the proportionate length of the bridge covered by the heavier locomotives and their tenders is greater, until when the span is only equal to the length of two locomotives and tenders, it would necessarily at times be entirely covered by this heavier load, and still shorter spans would be entirely covered by the heavier weighted wheels of the locomotives.* First find the panel load—*i.e.* the equivalent uniform load per foot run multiplied by the panel length = Q say. To find the maximum flange stresses assume the load Q to act at each joint on the upper or lower flange according as the load is carried on one or the other. Thus the girder will be designed for panel loads of the equivalent uniform load and not for the uniform load itself; in other words, for the equivalent uniform load preceded by a concentrated load equal to half the panel load. The stresses induced in the flanges by the load Q at each joint on the lower or upper flange, whichever is loaded, will be exactly the same as in the case of the dead load, taking P or P^1, as the case may be, equal zero, and substituting Q for the remaining one. This enables the maximum stress in the flanges due to the live load to be immediately found. As already shown, in the case of a brace the stress will be greater when the loads are taken at the panel points on the longer segment of the bridge only, and its vertical component obviously equals the shearing force on the shorter segment; as this segment is unloaded, as far as live load is concerned, the shearing force for that segment will be constant and equal to the reaction at that end of the girder. Finding the maximum vertical component of stress in the inclined braces due to the live load therefore reduces to finding the reaction at the end of the shorter segment on one side of the brace considered, the joint at its lower end (when the

* *See* "Moving Loads on Railway Underbridges," W. B. Farr (*Proceedings Inst. C.E.*, vol. cxli., p. 2); also "Stresses in Lattice-girder Bridges," F. C. Lea (*Proceedings Inst. C.E.*, vol. clxi., p. 261).

inclined braces are in tension), and all the other joints on the longer segment being loaded by a load Q.

As an example, take an N girder with six bays (Fig. 38), with the live load carried on the lower flange. The maximum vertical component of tensile stress in the first inclined brace equals the reaction at the adjoining end, when all the joints to the left of this bay are loaded $= \frac{5\,Q \times 3\,b}{6\,b} = \frac{15}{6}\,Q$; the maximum vertical component of tensile stress in the second inclined brace equals the

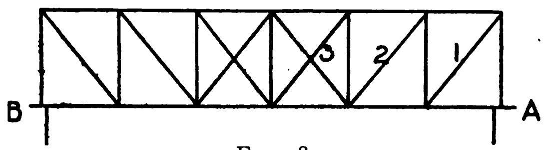


FIG. 38.

reaction at the same end as before when the joints to the left of this bay are loaded $= \frac{4\,Q \times 2\frac{1}{2}\,b}{6\,b} = \frac{10}{6}\,Q$; similarly, the maximum vertical component of tensile stress in the third inclined brace $= \frac{3\,Q \times 2\,b}{6\,b} = Q$; and the maximum vertical component of compressive stress in the fourth inclined brace $= \frac{2\,Q \times 1\frac{1}{2}\,b}{6\,b} = \frac{3}{6}\,Q$. When this compression is greater than the tension in the inclined brace, due to the dead load, it is usual to introduce a counterbrace to take the stress in tension instead of putting the brace into compression, which type of stress it is not designed to resist. In this case the maximum compression in the centre post, due to the live load, equals the maximum vertical component of tension in the counterbrace attached to it. It will be noticed that except for the end bay these values are greater than the corresponding stresses in Fig. 23, taking $P = 0$ and $P^1 = Q$.

The maximum stress in the vertical joined to an inclined brace at the opposite end to that to which the live load is applied will be equal to the vertical component of the maximum stress in this inclined brace—*i.e.* if the load is on the bottom flange the maximum stress in a vertical due to the live load will be the same as the maximum vertical component of stress in the brace attached to its upper extremity and *vice versâ.*

Next take an N girder with seven bays (Fig. 39), with the live

load carried on the lower flange. The maximum vertical component of tensile stress in the first inclined brace equals the reaction at the near end with the joints to the left of this bay loaded $= \frac{6 Q \times 3\frac{1}{2} b}{7 b} = 3 Q$; in the second inclined brace it equals $\frac{5 Q \times 3 b}{7 b} = \frac{15}{7} Q$; in the third inclined brace $\frac{4 Q \times 2\frac{1}{2} b}{7 b}$

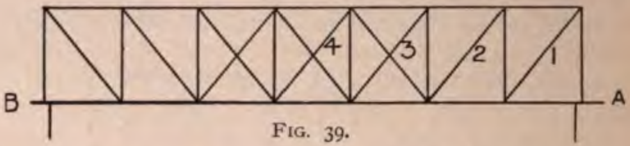


FIG. 39.

$= \frac{10}{7} Q$; in the fourth inclined brace $\frac{3 Q \times 2 b}{7 b} = \frac{6}{7} Q$; and the vertical component of the compression in the fifth inclined brace equals $\frac{2 Q \times 1\frac{1}{2} b}{7 b} = \frac{3}{7} Q$.

In this case the maximum stress in the fourth bay will obviously be the same whether the joints to the right only or to the left only of the central bay be loaded. When the first two joints from the left end only are loaded, there will occur the maximum compression in the fifth inclined brace, because both these loads put it in compression, and if this compression is greater than the tension in it, due to the dead load, a counterbrace would be introduced in that bay on either side of the centre. The maximum stress in the vertical joined to an inclined brace at the opposite end to that to which the live load is applied will be equal to the vertical component of the maximum stress in this inclined brace. The maximum compression in the two centre verticals due to live load will equal the maximum vertical component of stress in the middle brace, which is greater than the maximum vertical component in the counterbrace in the fifth bay.

If the bridge be erected before the counterbrace is inserted there would be no stress in it when the brace is in tension due to the dead load, therefore the counterbrace will be put into tension immediately the live load tends to reduce the tension in the brace—*i.e.* the maximum tension in the counterbrace will be equal to the maximum compression which the live load tends to produce in the brace. But if the counterbrace is attached during erection, so that there would be no stress in either brace

or counterbrace at the time, it is obvious that when the support is removed the brace comes into tension, due to the dead load, and the counterbrace can only come into tension after the live load has counteracted the tension in the brace due to dead load, whence it follows that the maximum tension in the counterbrace is the difference between the compression the live load tends to cause in the brace and the tension which the dead load produces in it. To allow for the effect of any initial stress, it is often designed to be capable of taking as a tension the maximum compression in the corresponding brace due to the live load.

The above considerations enable the stresses due to dead load and the maximum stresses due to panel loads of equivalent uniform live load to be readily obtained.

As an example, consider an N-type girder bridge of 140 feet span, carrying two lines of rails on cross-girders on the lower flange; it is taken 25 feet deep to give even figures, and has seven bays (Fig. 40). The weight of the girders is taken as 0·8 ton per foot run of the bridge, and the platform load 0·7 ton

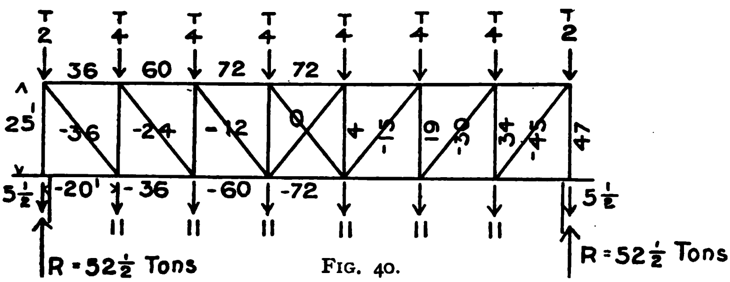


FIG. 40.

per foot run of the bridge; the wind bracing is taken in with the main girder; making a total dead load of 1·5 tons per foot. The live load on the bridge, carried on the lower flange, is such that the "equivalent" uniform load is 3 tons per foot run for the two lines of rails. To find the maximum stresses in the members—

The weight of the main girders per panel for each girder . . . $= \frac{0{\cdot}8 \times 20}{2} = 8$ tons.

The platform load per panel for each girder . . . . $= \frac{0{\cdot}7 \times 20}{2} = 7$ tons.

Total dead load per panel . 15 tons.

The weight of the girder is divided between the upper and lower panel points, giving 4 tons on each, and the 7-tons platform load is carried on the lower flange; thus the panel loads are, on the top flange 4 tons, and on the bottom flange 11 tons. Writing on the right side of the figure the shearing forces in each panel, which, as before, are the vertical components of the stress in the inclined braces, starting at the right support, these are 45 tons, 30 tons, 15 tons, and 0. The stresses in the corresponding vertical posts are 47 tons, 34 tons, 19 tons, and 4 tons.

The horizontal components of stress in these braces equal the vertical components multiplied by $\frac{20}{25} = \frac{4}{5}$, and are therefore, starting from the support as before, 36 tons, 24 tons, 12 tons, and 0. The stress in first length of upper flange is therefore 36 tons; in the second length 60 tons; in the third and fourth lengths 72 tons. In the lower flange the stresses have the same values as the upper flange between the same pair of braces. The horizontal stresses are written on the left half of the figure.

In this case the "equivalent" uniform live load equals twice the dead load—*i.e.* $Q = 2(P + P^1) = 30$ tons, therefore the maximum flange stresses due to the moving load equal twice those due to the dead load.

The maximum vertical components of tensile stress in the inclined braces, and therefore in the vertical posts connected to them at their upper extremities (since the load is carried on the bottom flange), starting from the right support, equal the reaction at that support when all the panel points to the left of the brace considered are loaded, which

in the first bay . . $= \frac{6 \times 30 \times 3\frac{1}{2}}{7} = 90$ tons,

in the second bay . . $= \frac{5 \times 30 \times 3}{7} = 64\frac{2}{7}$ tons,

in the third bay . . $= \frac{4 \times 30 \times 2\frac{1}{2}}{7} = 42\frac{6}{7}$ tons,

in the fourth bay . . $= \frac{3 \times 30 \times 2}{7} = 25\frac{5}{7}$ tons,

and the maximum vertical component of compressive stress

in the fifth bay . . $= \frac{2 \times 30 \times 1\frac{1}{2}}{7} = 12\frac{6}{7}$ tons.

The total stresses may now be arranged in tabular form.

Total maximum stress on the upper flange (tons):—

	Bay 1.	Bay 2.	Bay 3.	Bay 4.
Due to dead load . . .	36	60	72	72
Due to live load . . .	72	120	144	144
Total stresses	108	180	216	216

The stresses in the lower flange length between the same inclined braces will be the same as the above. In the end bay of the lower flange the stress due to the vertical loads is zero, but it has to transmit the stress due to the friction between the train and rails when the brakes are applied on the bridge, and is generally made the same section as the next bay for the sake of stiffness.

Total maximum stress in the verticals :—

	Bay 1.	Bay 2.	Bay 3.	Bay 4.
Due to dead load . . .	47	34	19	4
Due to live load . . .	90	$64\frac{2}{7}$	$42\frac{6}{7}$	$25\frac{5}{7}$
Total stresses	137	$98\frac{2}{7}$	$61\frac{6}{7}$	$29\frac{5}{7}$

Vertical component of maximum stress and total maximum stress in the diagonals :—

	Bay 1.	Bay 2.	Bay 3.	Bay 4.
Due to dead load . . .	45	30	15	0
Due to live load . . .	90	$64\frac{2}{7}$	$42\frac{6}{7}$	$25\frac{5}{7}$
Total vertical components .	135	$94\frac{2}{7}$	$57\frac{6}{7}$	$25\frac{5}{7}$
Total stress = last line × 1·28	172·8	120·7	74·1	32·9

The maximum vertical component of compression in the bay adjoining the central bay was found to be $12\frac{6}{7}$ tons, which is less than the vertical component of tension of 15 tons in the brace in that bay due to dead load, therefore these bays do not really require counterbracing, although it is a common practice to counterbrace one bay beyond the last one that really requires it.

(2) The "equivalent" uniform load method is, however, not entirely satisfactory, because although it gives the maximum flange stress, the same "equivalent" load does not necessarily give the maximum shearing force. A more satisfactory method is to take a typical load and find the maximum bending moment and shearing force due to any position of that load. The typical load need not, of course, be an actual load, but

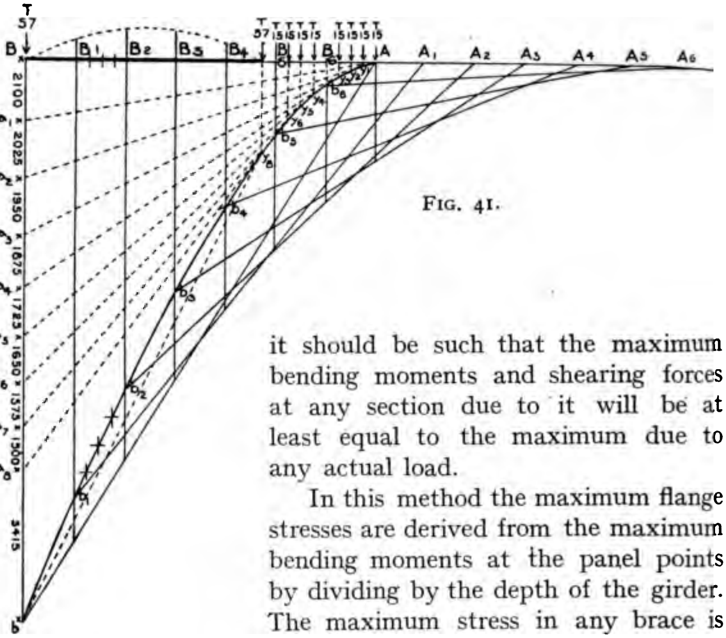


FIG. 41.

it should be such that the maximum bending moments and shearing forces at any section due to it will be at least equal to the maximum due to any actual load.

In this method the maximum flange stresses are derived from the maximum bending moments at the panel points by dividing by the depth of the girder. The maximum stress in any brace is best arrived at by finding the maximum shearing force in that bay for any position of the live load.

The method is best explained by taking an example. Suppose a girder of the same span and with the same number of bays as in the last case be taken; and let the typical load consist of: four axles each carrying a load of 15 tons and 5 feet apart, then a space of 10 feet followed by four more axles,

also loaded to 15 tons and 5 feet apart, representing two locomotives, and 5 feet behind the last axle a uniform load of 1·2 tons per foot run for the remainder of the length of the girder, as indicated in Fig. 41. Suppose the leading axle to be over the right support, the whole of the span thus being covered with load. First draw the bending moment diagram for this position of the load, plotting it below the line A B, as explained in the chapter on Bending Moments, page 12.

That is to say, from B plot vertically downwards the moments of each of the concentrated loads taken successively, starting at A, and substituting for the uniform load in the first instance a concentrated load at its two ends equal to one-half the uniform load. Draw the parabola of bending moment for a uniform load on a span equal to its length, on the corresponding part of the diagram, and finally add the ordinates of the parabola to the corresponding ordinates on the diagram of the bending moments of the concentrated loads. The distance from the leading axle to the front of the uniform load is 45 feet, therefore the length of this load is $140 - 45 = 95$ feet. Its weight is $95 \times 1{\cdot}2 = 114$ tons, and half of this (57 tons) is taken as acting at each end of its length—*i.e.* at 45 feet from the right support, and over the left support. The moments of the concentrated loads about B are :—

tons *feet* *tons-feet*

$$15 \times 140 = \frac{4200}{2} = 2100 = \text{B}\,a_1$$

$$15 \times 135 = \frac{4050}{2} = 2025 = a_1\,a_2$$

$$15 \times 130 = \frac{3900}{2} = 1950 = a_2\,a_3$$

$$15 \times 125 = \frac{3750}{2} = 1875 = a_3\,a_4$$

$$15 \times 115 = \frac{3450}{2} = 1725 = a_4\,a_5$$

$$15 \times 110 = \frac{3300}{2} = 1650 = a_5\,a_6$$

$$15 \times 105 = \frac{3150}{2} = 1575 = a_6\,a_7$$

$$15 \times 100 = 1500 = a_7\,a_8$$

$$57 \times 95 = 5415 = a_8\,b$$

Thus the moment of the reaction at A about B $= 19815$ tons-feet.

The centre ordinate of the parabola over the uniform load

$$= \frac{(95)^2 \times 1 \cdot 2}{8} = 1{,}354 \text{ tons-feet.}$$

Join a_1 to A and let this line intersect the vertical line through the next load in y_1; join a_2 to y_1 and let this line intersect the vertical through the next load in y_2; join a_3 to y_2 and let this line intersect the vertical through the next load in y_3, and so on.

Finally, join y_8 to b, and above this line plot vertically the ordinates of the parabola for the uniform load indicated in the figure. Then the vertical ordinates of the resultant curve $b\ b_1\ b_3\ b_6$ A is the bending moment diagram referred to the base A b. Vertical lines are drawn in the figure through the panel points of the girder intersecting the bending moment curve in the points $b_1\ b_2\ b_3\ b_4\ b_5\ b_6$.

Suppose, now, the leading axle moves a panel length back along the girder, then $A_1\ B_1$, in Fig. 41, would represent the position of the girder relatively to the load, and $b_1\ b_3\ b_6$ A A_1 (the ordinates being measured vertically as before) will be the bending moment diagram referred to $A_1\ b_1$ as base, for this position of the load, because the intercepts on $B_1\ b_1$ are obviously the moments of the loads about B_1—*i.e.* the moments about one support when the leading axle is one panel distant from the other support; $B_1\ b_1$ will therefore be the moment of the reaction at the right support for this position of the load. Similarly, if the leading axle moves back two panel lengths from the right support, the intercepts on $B_2\ b_2$ are obviously the moments of the loads about the left support for this position of the load, and $b_2\ b_3\ b_6$ A A_2 is the bending moment diagram referred to $b_2\ A_2$ for this position of the load, and $B_2\ b_2$ must be the moment of the reaction at the right support about the left support. Similarly when the leading axle moves back 3, 4, 5, and 6 panel lengths from the right end. In the last case b_6 A A_6 is the bending moment diagram.

To find the maximum bending moment, say at the first panel point from the left support, it is only necessary to measure the ordinate at the first panel point from the end on this series of bending moment diagrams on bases b A, $b_1\ A_1$, $b_2\ A_2$ . . . $b_6\ A_6$, and it will be found that the maximum occurs for the bending moment diagram on base $b_3\ A_3$—*i.e.* when the leading axle is three panels distant from the right support, when it measures 2,100 tons-feet; the second panel point from the left the maximum occurs

for the same position of the load, and measures 3,700 tons-feet; at the third panel point from the left the maximum occurs for the bending moment diagram on base $b_2 A_2$—*i.e.* when the leading axle is two panels distant from the right support and it measures 4,600 tons-feet; the maximum at the fourth panel point from the left is for the same position of the load, and measures 4,700 tons-feet; the maximum at the fifth panel point from the left is when the leading axle is one panel length from the right support, and measures 4,000 tons-feet; and at the sixth panel point the maximum also occurs for this position of the load, and measures 2,500 tons-feet. It will be noticed that the latter three are greater than the former three, and are therefore the maximum bending moments at the three panel points from either end; in other words, the maximum bending moment at a section here occurs when the load is approaching the end of the bridge nearer to which the section is taken. Of course, the bending moments might be rather greater for intermediate positions, but in most cases it will be seen that near the maximum the variation is slight. To find the maximum stresses in the inclined braces;—it has just been pointed out that $B_1 b_1$, $B_2 b_2$, . . . $B_6 b_6$ are the moments of the reaction at A, when the leading axle is 1, 2, . . . 6 panels lengths from A. Now when the leading axle is at the second panel point from the right support, the shearing force in the second bay is nearly a maximum; it will be actually a maximum when the leading axle has advanced some distance into that bay. When the leading axle is at that panel point the shearing force in the second bay equals the reaction at A $= \frac{B_2 b_2}{140} = \frac{13{,}200}{140} = 94$ tons by measuring $B_2 b_2$ on the diagram. When the leading axle is half-way across that bay the shearing force in the bay equals the reaction at A minus the portion of the load in the second bay transmitted to the cross-girder at its right extremity. The moment of the reaction at B is now equal to the ordinate one-half a bay length to the left of $B_2 b_2$, which measures 15,400. The reaction therefore equals $\frac{15{,}400}{140} = 110$ tons-feet. Now there would be two axles in the second bay, for in addition to the leading axle at the centre, the next axle comes in 5 feet to the left of the centre. The load transmitted to the cross-girder at the right extremity of the bay now equals $\frac{15}{2} + \frac{15}{4} = 11\frac{1}{4}$ tons; therefore the shearing

force in the second bay equals 110 − 11 = 99 tons. This is seen to be greater than when the leading axle is at the panel point. Examine what its value would be if the leading axle is one-quarter across the bay. To obtain the reaction we must measure the ordinate one-quarter bay length to the left of $B_2 b_2$, which scales 14,300 tons-feet, and the reaction therefore equals $\frac{14,300}{140} = 102$ tons. There will now be only one axle on this bay, and that at one-quarter of its length, which will bring a load on to the cross-girder at its right extremity equal $\frac{15}{4} = 3\frac{3}{4}$ tons. Thus the resultant shearing force is 98 tons, which is less than its value when the leading axle is at the centre of the bay. Next inquire whether the shear in this bay would be greater or less if the leading axle is at three-quarters the length of the bay from its left end. The moment of the reaction is now the ordinate at a distance of three-quarters a bay length from $B_2 b_2$, and measures 16,500 tons-feet, and the reaction is $\frac{16,500}{140} = 118$ tons. There would be three axles in the bay at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ its length respectively, and the load transmitted to the cross-girders at its right extremity equals $15\left(\frac{3}{4} + \frac{1}{2} + \frac{1}{4}\right) = 22\frac{1}{2}$ tons. The shearing force in the bay is therefore 118 − 22 = 96, which is also less than the value with the leading axle at the centre, which is evidently not far from the maximum, and at all events near enough for practical purposes.

The maximum shearing stress in any bay can be found in a similar manner, and thence the maximum stresses in the inclined braces and in the verticals.

Wind Pressure.—The actual wind pressure which should be allowed for on any structure such as a bridge, depends upon the exposure of the site of the structure, and the greatest pressure liable to occur at the site should always be provided for. In experimentally determining such intensity, the surface experimented upon should not be too small; the larger the better. The reason for this is that the pressure of the wind, especially when great pressures are developed, is seldom of equal intensity over a considerable area, the gusts being more intense at certain points than at others, so that it is seldom necessary for safety to provide for the maximum intensity of pressure, measured on a small surface, occurring on the whole extent of the structure at the same instant.

Dr. T. E. Stanton, in his paper on "Resistances of Plane Surfaces in a Uniform Current of Air," * arrived at the following conclusions for small models:—That the intensity of pressure at the centre of the windward side of a plane surface for a velocity of the wind of V feet per second is given by the formula $\frac{P}{G} = \frac{V^2}{2g}$, where P is the pressure in pounds per square foot, G the weight of 1 cubic foot of the air in pounds, and g the acceleration of gravity in foot-seconds per second; that the intensity of suction on the leeward side is practically uniform, except near the edges, but that its value varies with the form of the surface; for similar plates and similar lattice models the mean intensity of pressure is the same under similar conditions; that the resultant pressure P, of the wind in pounds per square foot, on small surfaces, is given by the formula $P = 0{\cdot}00126\,V^2$, but later experiments on surfaces over 1 square foot in area give the formula $P = 0{\cdot}0015\,V^2$; that for small lattice models the intensity of pressure is 25 per cent. higher than for square plates; that the mean resultant intensity of normal pressure on rectangular plates varies considerably with the ratio of length to width, increasing as this ratio increases; that when two similar circular plates are placed parallel to each other, near together, the resultant pressure is less than that on a single plate, when the distance apart is about 1½ diameters it is a minimum, at 2 diameters apart it is the same as for the single plate, at 5 diameters apart it is 1·8 times that on a single plate; that a somewhat similar effect is produced with a pair of similar or dissimilar lattice models; that for oblique currents on rectangular plates the resultant pressure depends upon whether the long or short axis is inclined to the current, for angles up to about 65 deg. between the direction of the wind and the normal to the plate the pressure is greater in the former case and for larger angles in the latter case; that for all cases of similar combinations of flat surfaces similarly situated the intensity of normal pressure at corresponding points is the same for the same velocity of current; that when two parallel girders connected with a roadway are at a distance apart equal to the depth of the girders the pressure on the leeward girder is 15 per cent. of that on the windward girder, and at twice that distance 25 per cent., the effect of the roadway being to reduce the resultant pressure on the girders, the resultant

* *Proceedings Inst. C.E.*, vol. clvi., p. 90.

normal pressure is slightly increased for small angles of obliquity of the wind. Some further results are referred to in connection with roofs.

The wind pressure acting normally to a bridge has the effect of altering the stresses in the flanges, which are also the flanges of the horizontal wind girders. The flanges which carry the load are braced together by the platform, and the other flanges are, when possible, connected by wind bracing. The girder on the windward side would be exposed to the full force of the wind; when the platform is carried on the lower flanges—*i.e.* when the bridge is a through bridge—the area it acts upon equals the length of the bridge multiplied by the maximum height of the train, say 13 feet 6 inches, and the exposed area of the members above and below this. If the depth of the girder be less than the height of the train, the wind pressure must of course be taken as acting on the full depth of the train and the area of the bridge below it. If the platform is carried on the upper flange—*i.e.* when the bridge is a deck bridge—the wind would, of course, act on the vertical area of the train plus the exposed area of the members of the girder for its full depth. Sometimes the exposed area of the bridge members above and below the train is doubled to allow for the action of the wind on the leeward side. In many cases this would differ little, in the case of a through bridge, whose depth is greater than that of the train, from taking the wind as acting on an area equal to the total length of the bridge multiplied by its total height. The maximum intensity of the wind pressure is generally taken between 30 and 56 pounds per square foot, according to the exposure of the site. The wind load is sometimes taken as a live load, and sometimes as a dead load, and at times partly one and partly the other; but if the intensity of wind pressure be taken high enough, having regard to the fact that actually the wind pressure is never constant over the area of the bridge, it would appear to be sufficient in most cases to take it as a dead load.

In the case of a through bridge, the two lower flanges and the platform between them, and the two upper flanges and the wind bracing between them, constitute two horizontal girders, each of which, when the main girders are of the parallel type, would take about one-half the total wind pressure, which may be taken in the calculations as loads concentrated at the panel points. For the reason given below it is best to assume that

the lower flanges and the platform bracing take, say, three-fifths of the wind pressure. In a deck bridge the upper flange would, of course, have to take up most of the wind pressure.

Since the upper flanges are in compression that compression would be increased by the wind when it blows from the side of the flange under consideration; and since the lower flanges are in tension that tension would be increased when the wind blows from the opposite side of the bridge to that of the flange considered. If, therefore, we take a girder (Fig. 42) in which the distance from centre to centre of the flanges equals the horizontal distance from centre to centre of the two main girders of the bridge, and take loads acting at each panel point in the

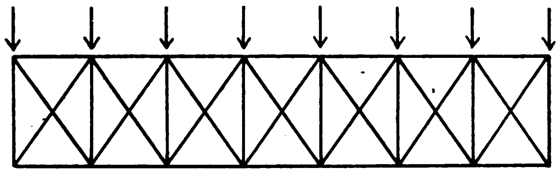


FIG. 42.

same way as when the stresses due to dead load in the vertical girder were calculated, then the stresses in the flange of such girder on the windward side will represent the maximum increase of compression in the top flange due to the wind, and the stresses in the flanges on the side away from the loads will be the maximum increase of tension in the lower flanges due to the wind, and the stresses in the bracing would be the stresses in the wind bracing. Each horizontal bay is counterbraced because sometimes the wind blows from one side of the bridge and sometimes from the other side. In a through bridge the wind stresses in the upper flange have to be transmitted to the supports; this is partially effected by transferring part of the wind load to the lower flange at each vertical, which is a reason for taking a larger proportion of the wind pressure as being taken up by the lower flange; but in the calculations it may be taken as being transmitted by the end members, which are often stiffened for the purpose by an overhead portal arch.

To avoid overturning of the bridge due to wind pressure, it is obvious that the weight of one main girder multiplied by the distance apart of the girders plus the weight of the platform, etc., multiplied by one-half that distance, must be greater than

the total maximum pressure of the wind multiplied by the height of its centre of pressure above the supports. A factor of safety of 2 is ample in this case.

As examples of the graphical method of finding the stresses in braced parallel girders, take first the N-type girder as shown by full lines in Fig. 43. In this and the following diagrams the

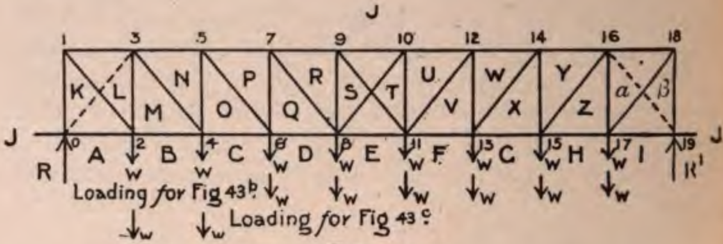


FIG. 43.

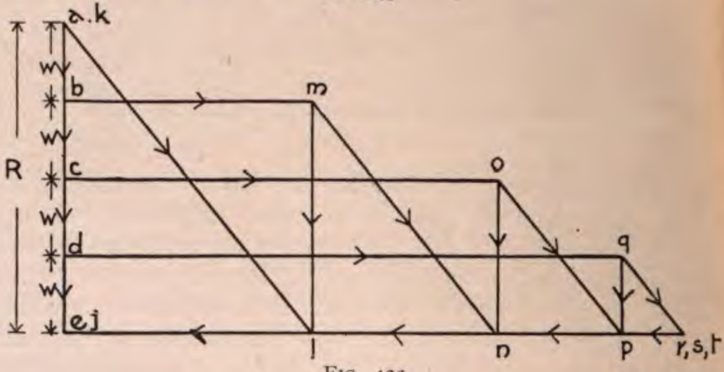


FIG. 43*a*.

portion of the load from the end bays directly transferred to the abutments is omitted, so that R and R^1 are the reactions for the loads at the intermediate panel points only. Place letters in the spaces between the applied forces and between the members of the structure, so that each load and the stress in each member is denoted by the two letters on either side of it. Also number the joints in the order that their equilibrium will be considered—*i.e.* such that there will not be more than two forces of unknown magnitude acting at any joint when its turn to be considered arrives.

Fig. 43*a* is a stress diagram for the N girder with nine bays (Fig. 43), when all the lower joints are loaded with an equal load W. When W is taken equal to the panel dead load, the

stresses in this diagram are the dead load stresses; and when W is taken equal to the panel live load (when the live load covers the bridge), Fig. 43*a* would give the maximum flange stresses due to live load.

First draw the load line *a b c d e*. As the joints are symmetrically loaded, the stress diagram for the right half of the girder will be the same as for the left half, and *e a* will be the reaction at the left support. Draw the polygon of forces for each joint in the order numbered in Fig. 43, taking the forces and stresses acting on a joint as they come in order, following round the joint in the contra-clockwise sense—*e.g.* the letters round the joint 1 are J K L J, therefore J *k l* J in Fig. 43*a* is the triangle whose sides are parallel and proportional to the forces and stresses acting on joint 1, and is therefore the triangle of

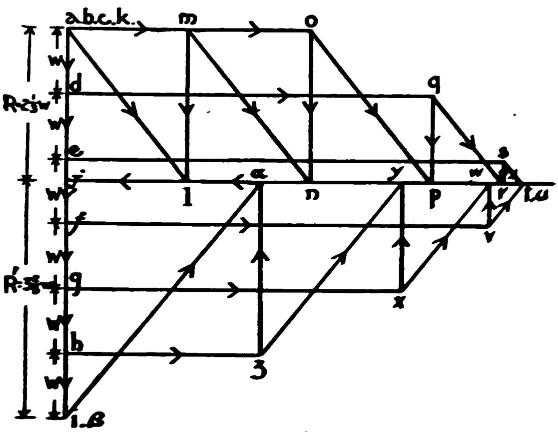


Fig. 43*a*.

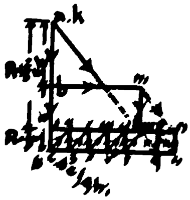


Fig. 43

forces for that joint. At joint 2 [illegible] rounding the joint are [illegible] in Fig. 43*a* is therefore [illegible] in a member on a joint in Fig. [illegible] the line in Fig. 43*a* [illegible] order the spaces are taken: [illegible] wards, therefore the [illegible] that joint—*i.e.* [illegible] compression [illegible] are M X J L M, therefore [illegible] at that joint; and since, [illegible] the stress in M X in [illegible]

and is therefore a tension. In the same way joint 4 is surrounded by the spaces denoted by the letters N M B C O N, therefore the corresponding polygon *n m b c o n* is the polygon of forces for that joint, and so on. It will be noticed that the arrows in Fig. 43*a* refer in each case to the first of the two joints connected by any given member. To determine the nature of the stress in any member it is only necessary to notice the order of the letters on the two sides of it, taken in the contra-clockwise sense with reference to the joint at either end, and the direction of the line represented by the same two letters in the same order in Fig. 43*a* gives the direction of the stress in the member on the joint considered, which at once shows whether the stress is a tension or a compression.

If a section be taken through any series of members, the stresses in these members must be in equilibrium with the loads and the reaction on either side of that section, and it will be seen by trying any particular case that the stresses in the severed members and the external forces on either side constitute a closed polygon in Fig. 43*a*. The same consideration, of course, applies in all cases; *e.g.* if a vertical section is taken through the third bay the members severed are J P, P O, O C, and *j p o c b a j* is the closed polygon.

Fig. 43*b* is the stress diagram with the six joints adjoining the right end only, loaded as indicated in Fig. 43. This is the position of the live load that would give a maximum tension in O P and a maximum compression in O N. In order to determine this maximum stress it is not necessary to draw the whole diagram, but it is shown complete in Fig. 43*b* in order to indicate the stress in the other members with this load. In this case the brace S T between the joints 9 and 11 is in tension, as seen from Fig. 43*b*; therefore the brace between the joints 8 and 10 will be out of action. Fig. 43*c* gives the stress diagram when the two joints adjoining the left end only are loaded, as indicated in Fig. 43. This would be the position of the live load that would give a maximum compression in the brace O P, and the maximum tension on the post O N; the brace S T between the joints 8 and 10 is now in tension, therefore the brace between joints 9 and 11 is out of action. In this case again it is not necessary to draw the whole diagram to find these maximum stresses, but it is here completed to indicate the amount of the stresses in the other members for this position of the load.

Fig. 43*d* gives the stresses in the Pratt girder denoted by

the dotted diagonals in Fig. 43, the end verticals and top flange members being omitted; it will be seen that the values of the stresses in the members is unaltered except that the flange member between joints 0 and 2 has now the same stress as that borne by the flange member between joints 1 and 3 in the N-type girder.

In Fig. 44 a girder with all the braces inclined is shown. Fig. 44*a* is the stress diagram when all the joints are loaded, giving the dead load stresses when W is the panel dead load, and the maximum flange stresses due to live load when W is the panel live load.

Fig. 44*b* is the stress diagram when the four joints adjoining

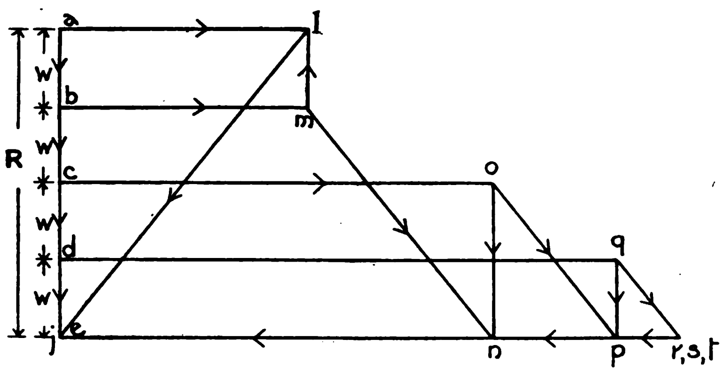


FIG. 43*d*.

the right support only, as indicated in Fig. 44, are loaded by the moving load. Fig. 44*c* is the stress diagram when the two joints adjoining the left support are loaded by the moving load; the former case gives the maximum tension due to live load in the brace M N, and the maximum compression in the brace L M; and the latter gives the maximum compression due to live load in the brace M N, and the maximum tension in the brace L M.

Further stress diagrams are given in Chapter IV.

In designing a girder it is most important to so choose the shape of the flanges, inclined braces, and vertical or inclined struts that their axial lines through their centres of gravity may intersect at the panel points, in order to avoid secondary stresses due to excentricity; otherwise, the stresses would not be uniformly distributed over their cross-sections, and the material

would not be used to the best advantage. The upper flange being in compression must be of such shape that its radius of gyration about both vertical and horizontal lines through the centre of gravity of its cross-section shall be as large as possible, to avoid any considerable increase of area due to applying the column formula. When the flange area required warrants its use a trough section is most suitable for this purpose, con-

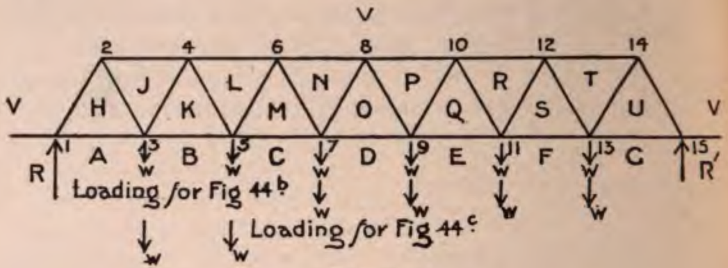


FIG. 44.

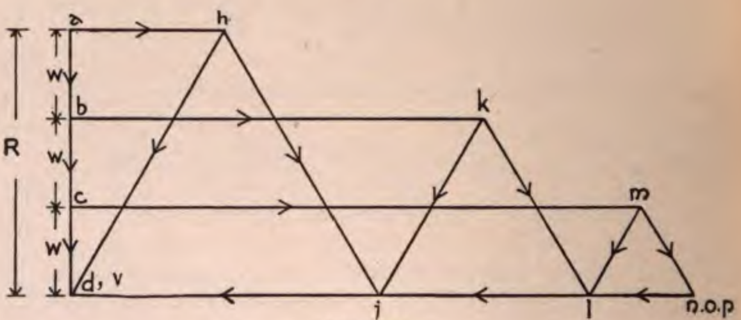


FIG. 44*a*.

sisting of a single plate running along the top with side plates connected thereto by angles (Figs. 45 and 46), with angles riveted to the lower end of the vertical plates for stiffening purposes and to keep the centre of gravity near the middle of the section. In the lower flange the horizontal plate would be beneath; it is sometimes omitted, but it serves to stiffen this flange when used. The vertical post must fit between the vertical plates of the flange, in large spans, with a connecting plate between it and them (Figs. 46 and 47) to secure the inclined ties to. The verticals may consist of two channel sections or be of an **I** section, consisting of plates and angles connected

together with trellis bars or a web plate (Figs. 45 and 47). These, being in compression, have to satisfy the column formula, taking the radius of gyration about the axis which gives a minimum result. The braces each consist of two plates, one connected to either side of the flange. In large spans, connecting plates

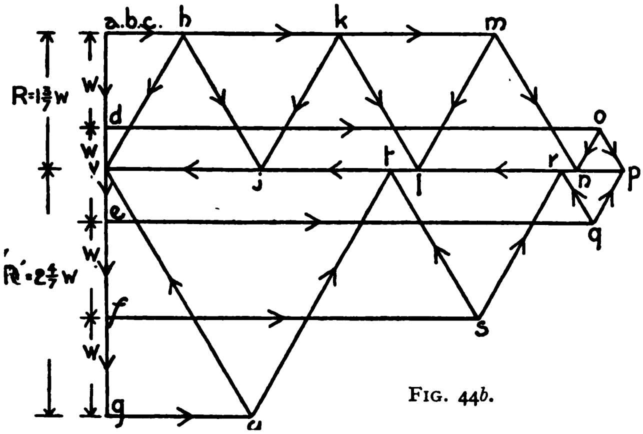


FIG. 44*b*.

above referred to are riveted inside the flanges, and the braces connected to these plates; this gives sufficient room for the number of rivets necessary to connect the verticals and inclined braces to the flanges. A number of the rivets will be common to the vertical, connecting plate, and flange, and it will be seen that fewer rivets are required through the flange with the connecting plate than without it, because the brace and vertical are first secured to the connecting plate, which is itself attached to the flange, thus the stress to be communicated from the connecting plate to the flange is less than the stress in the brace alone, being the resultant of that tension and the compression in the vertical connected to it. The plates in the braces, except when they are quite small in section, may be kept in the same plane as the connecting plates, and butt-strips used to make the joints, the rivets, of course, being in

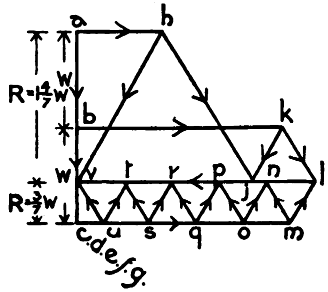


FIG. 44*c*.

double shear, the number of rivets multiplied by twice the cross-sectional area of a rivet and by the unit shearing stress being equal to the total stress in the braces, unless the plates are thin enough to necessitate the number of rivets being proportioned for bearing. The rivets connecting the verticals to the connecting plates are in single shear, therefore the number of rivets multiplied by the cross-sectional area of a rivet and by the unit shearing stress must equal the total stress in the vertical.

The flange requires increasing in cross-section as the centre is approached, due to the increasing stress; it is best to add additional side plates for this purpose, and not additional top plates, in order that the intersection of the centre lines of the braces and verticals may lie on the centre of gravity of the cross-section of the flange. When pin joints are used to connect the braces and verticals to the flanges, as is common in American practice, in some cases additional vertical plates running longitudinally are introduced in the centre of the flanges. The additional side plates have necessarily to be placed inside the flange, thus the width of the verticals must slightly decrease towards the centre of the bridge; but this is not of importance, because their requisite cross-sectional area decreases with the diminishing shearing force towards the centre.

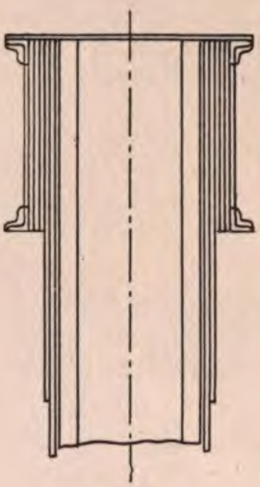


FIG. 45.
Section at A B, Fig. 46.

In arranging the cover plates to connect the braces to the connecting plates and to connect two lengths of plate in the flanges, it is best to have only one or two rivets in the outside rows, and gradually increase the number up to the joint, as shown in Fig. 46. This avoids weakening the plates connected by more than about two rivet holes. However, the cover plates will be weakened by the full number of rivets in the middle rows, as seen in Fig. 46, as the entire stress has to be transmitted through the cover plates at that section. This explains why the cover plates should be rather thicker than the plates connected.

The additional side plates in the flange required in the bay

to the right of the vertical in Fig. 47 may be extended to the left of the vertical to act as a cover plate to join up separate lengths of plate in the flange. The join in the outside plate is shown as coming farthest from the vertical and an outside cover plate is shown to cover it; cover angles would also be required, and the joint in the angles must not coincide with that

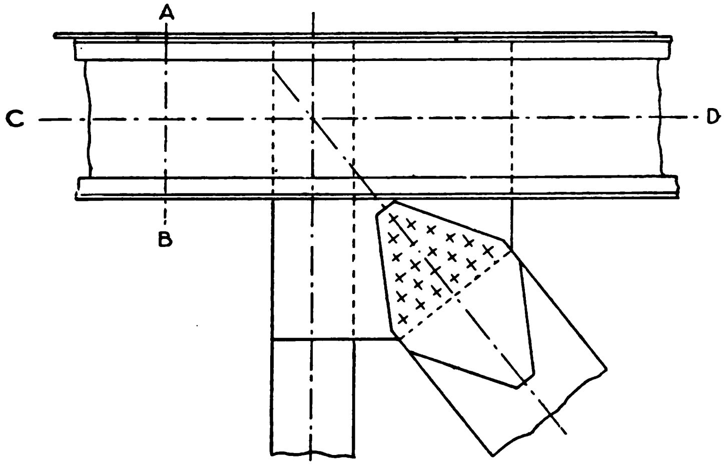


FIG. 46.

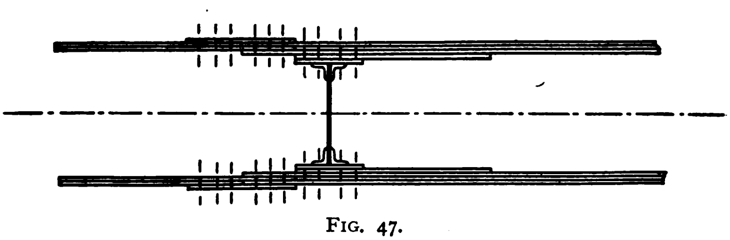


FIG. 47.
Section at C D, Fig. 46.

in a plate. The rivets in this case must be calculated for single shear, and the cover plates must be rather thicker than any of the plates they connect. Even if the cover plates are extended on both sides over the whole length of the joins in the side plates of the flange, since these plates are thin, the rivets would have to be calculated for bearing pressure, and the number of rivets required would not generally be much reduced — ·sual to increase the number of the rivets which ha· in

during erection by about 10 per cent., unless that riveting can be done by hydraulic or pneumatic riveters.

In the end bays of the bridge the plates forming the sides of the verticals should be kept wide enough to allow of a second row of rivets outside the angles, otherwise there will be a difficulty in getting in a sufficient number of rivets to secure the verticals to the connecting plates.

Since the cross-girders are connected to the feet of the verticals, it must be remembered that the rivet-holes to connect the cross-girders to the verticals have to coincide with the rivet-holes to connect the side plates of the verticals to their angles.

The end members, whether vertical or inclined, generally correspond in cross-section to the upper flange, as this section makes a good finish and enables the additional cross-sectional area required for the greater shear at the ends to be readily provided, besides giving the transverse stiffness necessary to transmit the wind pressure from the upper flange to the supports.

In bridges under 200-feet span one end is generally fixed longitudinally, but supported on bearings with cylindrical upper surfaces, so that the bridge can sag in the vertical plane without thereby introducing stresses. At the other end the bridge rests on the cylindrical surface of a rocker, the lower surface of which is also cylindrical and rests in a socket in the bed-plate shaped to receive it. This enables the bridge to expand and contract, and to sag without thereby introducing stresses.

For larger spans a series of rollers are used instead of a single rocker for the purpose of allowing for expansion and contraction, but the cylindrical joints to permit of sag are retained. The maximum permissible pressure on roller bearings in pounds per lineal inch in such a position, between parallel plates, is often taken as $500\sqrt{d}$, where d is the diameter of the roller in inches. In the case of a rocker-bearing in which the ends of the rocker are turned to fit bored sockets, the intensity of compression in the rocker, found by dividing the weight transmitted through it by the horizontal projection of the area in contact with the sockets, should not exceed about 2 tons per square inch, if it is possible to so arrange.

CHAPTER IV

BRACED GIRDERS—CURVED TYPE, AND ROOFS

I.—*Parabolic Girders*

IT has already been pointed out that if instead of the depth D being constant in the formula $M = H \times D$, H is constant, then D must vary as the bending moment; consequently, if the girder is to carry a uniformly distributed load, it will be parabolic in shape, *i.e.* the upper panel points will lie on a parabola. When such a girder is subjected to a moving load, H would no longer be constant except the load be uniform and covers the bridge, but would have a value in the flange in each bay length, which can be calculated for any given position of the load. In this case it is easiest to determine the stresses in the flanges from the bending moments at the panel points.

It is easier to understand the method if an actual example be taken than by taking the general çase, but the propositions which are proved for the particular case considered apply generally.

From the property of the parabola that the ratio of the ordinates at two points equals the ratio of the products of the segments into which these ordinates divide the span, the ordinate at any panel point when the central depth is known can be readily found. This property is easily recollected from the formula for the bending moment at the distance x, from the end of the span l, when subjected to a uniform load of intensity w—viz., $M = \dfrac{w\,x\,(l - x)}{2}$.

Suppose the span of the girder to be 120 feet, and its centre depth one-fifth of the span = 24 feet, and let it be divided into six bays. Call the depths at the panel points (Fig. 48) D_1, D_2, and D_3, then we have

$$\frac{D_1}{D_3} = \frac{1 \times 5}{3 \times 3} \qquad \text{or } D_1 = \tfrac{5}{9} \times 24 = 13\tfrac{1}{3} \text{ feet,}$$

and $$\frac{D_2}{D_3} = \frac{2 \times 4}{3 \times 3} \qquad \text{or } D_2 = \tfrac{8}{9} \times 24 = 21\tfrac{1}{3} \text{ feet.}$$

The length of $AC^1 = \sqrt{\left(\frac{40}{3}\right)^2 + 20^2} = \frac{20}{3} \times \sqrt{13} = 24$ feet.

The length of $C^1 D^1 = \sqrt{8^2 + 20^2} = 21{\cdot}5$ feet.

The length of $D^1 E^1 = \sqrt{\left(\frac{8}{3}\right)^2 + 20^2} = 20{\cdot}2$ feet.

Let the total dead load on the bridge be $1{\cdot}5$ tons per linear foot, and the equivalent live load considered as a panel load $3{\cdot}0$ tons per linear foot.

Then in each girder the dead panel load $= \frac{1{\cdot}5}{2} \times 20 = 15$ tons, and in each girder the live panel load $= \frac{3}{2} \times 20 = 30$ tons.

When the live load covers the bridge the total panel load = 45 tons.

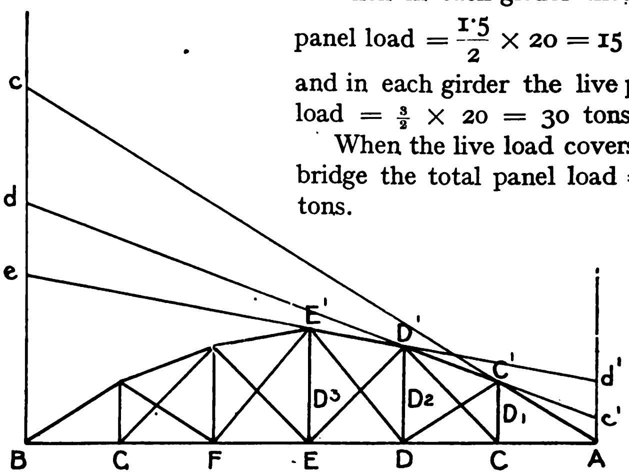


FIG. 48.

The maximum stresses in the flanges will occur as before, when the live load covers the bridge.

Then $H = \frac{M}{D} = 45 \times 6 \times \frac{120}{8 \times 24} = 168\frac{3}{4}$ tons; of which $56\frac{1}{4}$ tons is due to dead load and $112\frac{1}{2}$ tons is due to live load.

Therefore the maximum stress in the straight flange = $168\frac{3}{4}$ tons.

The maximum stress in $A\,C^1 = 168\frac{3}{4} \times \frac{24}{20}$ tons.

" " $C^1 D^1 = 168\frac{3}{4} \times \frac{21{\cdot}5}{20}$ tons.

" " $D^1 E^1 = 168\frac{3}{4} \times \frac{20{\cdot}2}{20}$ tons.

To obtain the maximum stresses in the inclined braces. There is no stress in those due to dead load. This is clear when it

is considered that the elevation of the girder represents the bending moment diagram for the dead load, and the shearing force in any bay equals $\frac{d\,M}{d\,x} = H\frac{d\,D}{d\,x} = H\frac{\delta\,D}{b}$, where $\delta\,D$ is the difference of the depth at two consecutive panel points and b is the panel length. Now $H \times \frac{\delta\,D}{b}$ is the vertical component of stress in the upper flange, which is thus equal to the shearing force and leaves no shear to be taken up by the braces. To find the maximum stresses in the braces due to concentrated loads equal to the panel load at the joints, these must be taken as before as acting at the panel point at the lower end of the brace in question and at all other panel points in the longer segment on whichever side of the brace it falls. The horizontal component of stress in the brace will then be equal to the difference of the horizontal components of the stresses in the two lengths of the flange meeting at its upper end. To obtain the maximum horizontal component of stress in the brace $C^1\,D$ in the second bay, find the reaction R_{A2} at A due to a load of 30 tons at the joints D, E, F, and G.

$$R_{A2} = \frac{4 \times 30 \times 2\tfrac{1}{2} \times 20}{120} = 50 \text{ tons.}$$

Calling H_1, H_2, and H_3 the horizontal components of stress in the upper flange, and M_1, M_2, and M_3 the bending moments, at the first, second, and third verticals for any given condition of load. For the particular load in this case $M_1 = H_1 \times D_1$ and $M_2 = H_2 \times D_2$, therefore the maximum horizontal component of stress in the brace $C^1\,D = H_2 - H_1 = \frac{M_2}{D_2} - \frac{M_1}{D_1} = \frac{50 \times 40}{\frac{64}{3}}$

$$- \frac{50 \times 20}{\frac{40}{3}} = 93\tfrac{3}{4} - 75 = 18\tfrac{3}{4} \text{ tons.}$$

It will be noticed that $18\tfrac{3}{4}$ tons is one-sixth of $112\tfrac{1}{2}$ tons, the tension in A B due to the live load covering the bridge.

The maximum tension will occur in the brace $D^1\,E$ in the third bay when the joints E, F, and G are loaded; the reaction R_{A3} at A due to a load of 30 tons at each of these joints is

$$R_{A3} = \frac{3 \times 30 \times 2 \times 20}{120} = 30 \text{ tons.}$$

The maximum horizontal component of stress in the brace D¹ E with this load is

$$H_3 - H_2 = \frac{M_3}{D_3} - \frac{M_2}{D_2} = \frac{30 \times 60}{24} - \frac{30 \times 40}{\frac{64}{3}} = 18\tfrac{3}{4} \text{ tons.}$$

The maximum tension in the counterbrace C D¹ in the second bay will occur when the maximum compression would be caused in C¹ D if the counterbrace were absent—*i.e.* if the joint C only is loaded. Under these conditions the reaction R_{B1} at B due to a load of 30 tons at C is

$$R_{B1} = \frac{30 \times 20}{120} = 5 \text{ tons.}$$

The maximum horizontal component of tension in the counterbrace C D¹ with this load is

$$H_1 - H_2 = \frac{M_1}{D_1} - \frac{M_2}{D_2} = \frac{5 \times 100}{\frac{40}{3}} - \frac{5 \times 80}{\frac{64}{3}} = 18\tfrac{3}{4} \text{ tons.}$$

The maximum tension in the counterbrace D E¹, in the third bay, will occur when the maximum compression would be caused in D¹ E if the counterbrace were absent—*i.e.* if the joints C and D are loaded. Under these conditions the reaction R_{B2} at B due to loads of 30 tons at C and D is

$$R_{B2} = \frac{2 \times 30 \times 1\frac{1}{2} \times 20}{120} = 15 \text{ tons.}$$

The maximum horizontal component of tension in the counterbrace D E¹ with this load is

$$H_2 - H_3 = \frac{M_2}{D_2} - \frac{M_3}{D_3} = \frac{15 \times 80}{\frac{64}{3}} - \frac{15 \times 60}{24} = 18\tfrac{3}{4} \text{ tons.}$$

It is clear that if there is a tension in either the brace or counterbrace attached to the lower end of any vertical, that this tension will produce a compression in the vertical which will reduce the tension in it due to the dead and live load at its lower extremity, therefore the maximum tension will occur in the verticals when the live load covers the spans, and will equal 45 tons; and the minimum tension will occur in this case in D D¹, for instance, when the joint C only is loaded, and it will equal the dead load at D minus the vertical component of the tension in the counterbrace D E¹ $= 15 - 6\frac{1}{4} = 8\frac{3}{4}$ tons. For it will be noticed the tension in D E¹ with a load at C only is one-third what it is with loads at C and D, because the reaction at B in the former case is one-third of its value in the latter case.

t will be seen from the above that the maximum horizontal component of stress in the flanges equals the maximum stress in the straight member A B. The maximum horizontal components of stress in the braces and counterbraces are all equal to one-sixth (six being the number of bays) of the stress in A B due to the live load covering the bridge. If, therefore, in Fig. 48, A B be taken to represent the maximum stress in that member due to the live load, and perpendiculars are erected at A and B, and the flange lengths A C^1, C^1 D^1, D^1 E^1 are produced both ways so as to intersect these perpendiculars, then—

I.—The lengths of the lines A c, c^1 d, d^1 e intercepted between these perpendiculars represent the maximum stresses due to live load in the lengths of flange A C^1, C^1 D^1, and D^1 E^1 on the same scale that A B represents the maximum stress in itself due to live load, because A B is the horizontal projection of these lines.

Again, the bay lengths represent the maximum horizontal components of the stresses in the braces and counterbraces on the same scale as before, because their horizontal components equal the maximum stress in A B due to live load divided by 6, therefore—

II.—The lengths of the braces and counterbraces represent, on the same scale as before, the maximum stresses in them.

III.—The maximum tensions in the verticals equal the panel loads due to dead and live load together.

These properties are not dependent on a particular span, or the number of bays, or the depth adopted in working out this example, but are true generally whatever the span, number of bays, and the depth at the centre may be.

It can be proved in the general case, but the author considers it is more convincing for the student to work out for himself another example and convince himself that the same deductions apply in the case considered.

If a typical load be taken, as discussed in the case of parallel girders, the bending moments and shearing forces can be scaled from a similar diagram of moments to that before drawn (Fig. 41). For any other method of loading except that of equal panel loads above explained, the lengths of the braces and counterbraces, or of the flanges produced to the verticals through A and B, will of course no longer be a measure of the stress in them.

II.—*Curved or Bowstring Girders*

Sometimes the points of intersection of the curved flange lengths of a girder—*i.e.* the panel points in the flange referred to—lie on a curve which is not a parabola. In this case the method of proceeding is much the same as in the case of the parabolic girder, but of course the same relationships between

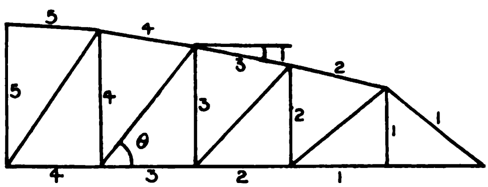


FIG. 49.

the stresses and lengths of the lines in the elevation will not hold in this case. This girder may, as before, be designed either for panel loads at the panel points, or for a typical load, the latter being generally more satisfactory.

The following is a convenient method of arranging the work in determining the stresses in such a girder due to dead and live loads. The stresses in the right half of the girder being those under consideration.

STRESSES IN FLANGES DUE TO DEAD LOAD (see Fig. 49).

Upper flange length numbered.	Depth at panel point, left end.	B.M. at panel point, left end.	$H=\frac{B.M.}{Depth}$.	Total stress in upper flange, H sec ι.	Lower flange length numbered.	Stress in lower flange.

The depth in column 2 and the bending moment in column 3 are taken at the end of the flange length nearer the centre, because there is no other inclined member under stress to the right of the vertical at that end.

The stress in the last column is that in the bay to the left of a vertical, and equals the horizontal component of the stress in the upper flange length to the right of the vertical.

STRESSES IN INCLINED BRACES DUE TO DEAD LOAD (see Fig. 49).

Inclined brace numbered.	H at vertical, left end.	H at vertical, right end.	Difference = horizontal component in inclined brace (B^1).	Total stress in inclined brace, $B^1 \sec \theta$.

STRESSES IN VERTICALS DUE TO DEAD LOAD (see Fig. 49).

Vertical to left of brace in last table.	$B^1 \tan \theta$ in brace, right side (*a*).	Load at foot of vertical (*b*).	Stress in vertical $= (a) \sim (b)$.

MAXIMUM STRESSES IN FLANGES DUE TO LIVE LOAD (see Fig. 49).

Upper flange length numbered.	Depth at panel point, left end.	Max. B.M. at panel point.	$H = \frac{\text{Max. B.M.}}{\text{depth}}$	Total stress in upper flange $H \sec \iota$.	Lower flange length numbered.	Stress in lower flange.

The depth in column 2 and the maximum bending moment in column 3 are taken at the end of the flange length nearer the centre, since there is no other inclined member under stress to the right of the vertical at that end. The stress in the last column is that in the bay to the left of a vertical, and equals the horizontal component of the stress in the upper flange length to the right of the vertical.

MAXIMUM STRESSES IN INCLINED BRACES DUE TO LIVE LOAD (see Fig. 49).

Inclined brace numbered.	Leading axle at vertical.	B.M. at vertical.		$\frac{\text{B.M.}}{\text{depth}}$ = H at vertical.		Difference = B = horizontal component in inclined brace.	Total stress in inclined brace (B sec θ).
		Left end.	Right end.	Left end.	Right end.		

The bending moments in columns 3 and 4 are to be measured on the bending moment diagram when the leading axle is at the lower end of the inclined brace considered. The

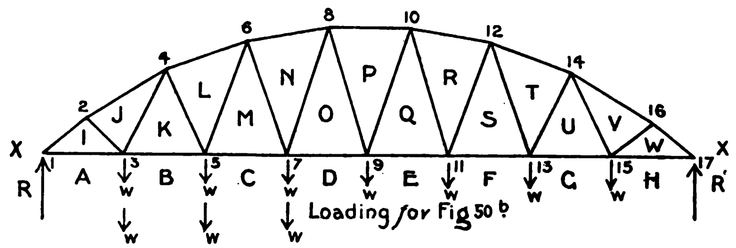


FIG. 50.

difference in the value of H at the verticals at the left and right ends will be rather greater when the leading axle has advanced into the bay, so that the values should be considered when the leading axle has advanced one-quarter of the bay length beyond the panel point, and when it has advanced one-half bay length beyond the panel point, as explained in connection with parallel girders; but in this case it is easier to work from the bending moment diagram than with the shearing force.

MAXIMUM STRESSES IN VERTICALS DUE TO LIVE LOAD (see Fig. 49).

Vertical to right of brace in last table.	H tan ι at vertical (a).	H tan ι at vertical to left (b).	B tan θ in brace, left side (c).	Stress in vertical (b) + (c) − (a).

The application of the graphic method to determine the stresses in the members of a curved girder due to dead and

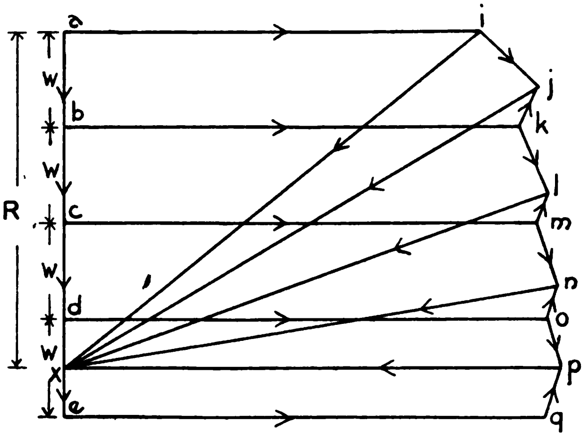


FIG. 50*a*.

live load is shown in Figs. 50, 50*a*, and 50*b*. Fig. 50 represents the elevation of a curved girder; Fig. 50*a* is the stress diagram for it when all the joints are loaded with a load w. If w is therefore the panel dead load, the stresses given by this figure are the dead load stresses; and if w is the panel live load, the stresses in the flange

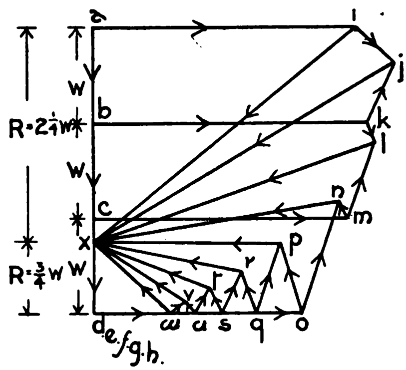


FIG. 50*b*.

members will be the maximum live load stresses. The method of drawing the stress diagram is the same as that described for the parallel girders, but in this case the lines in Fig. 50*a* representing the stresses in the upper flange members all radiate from x. The arrows in the diagrams, as before, give the direction in which the stress in the member acts upon the joint at its end denoted by the lower number in Fig. 50.

It will be observed, in Fig. 50*a*, that the stresses in the lower flange members are nearly constant—if the girder were parabolic they would of course be constant; also the stresses in the web members are comparatively small.

In Fig. 50*b* the stress diagram is given when the three joints adjacent to the left support only are loaded, and it will be noticed in this case that the tension in N O and the compression in O P are greater than the stresses in the previous case.

ROOFS

The loads that roof trusses are called upon to bear are (1) their own weight; (2) the weight of the roof covering; (3) in climates subject to snowstorms the maximum depth of snow that can adhere to the roof; (4) wind pressure and suction.

The weight of an iron or steel roof truss of moderate span may be roughly arrived at from the formula—

$$W = 0{\cdot}8\ B \times L\,(1 + 0{\cdot}1\ L),$$

where W is the weight of the truss in pounds, L is the span in feet, and B is the distance apart of the trusses in feet.

For wooden roofs the weight is about five-eighths of the result given by this formula.

The trusses or principals support, over or near the panel points in their upper chords, the longitudinal purlins of iron, steel, or wood. These in their turn support the rafters, fixed parallel to the top chords of the principals. When the outer covering is to consist of slates or tiles, the rafters are closely covered by boards fixed to the rafters, and the slates or tiles are nailed to the latter, preferably with copper nails, to prevent rusting. The total weight of the roof covering may vary between 10 and 30 pounds to the square foot, according to its nature. The weight of snow that may possibly cling to the roof depends on the locality and on the pitch of the roof, and its weight can be estimated from the fact that 1 cubic foot of snow weighs about 5 lbs.

The most indefinite portion of the load is that due to wind

pressure, and in the absence of sufficient experimental data it is impossible to dogmatise on the subject as to the exact distribution of wind pressure and suction; but one point is clear, which is, that there is no longer any excuse for assuming that the effect of the wind is confined to pressure on the windward side of the roof. Nor can experimental results regarding the pressure of the wind on plane surfaces be taken as an indication of the effect of the wind on a roof. Dr. T. E. Stanton, in his paper previously referred to on "The Resistance of Plane Surfaces in a Uniform Current of Air," * has very clearly established the fact that the pressure on a plane surface is made up of the pressure on the windward side and the suction on the leeward side, which latter depends on the shape of the surface experimented upon. He also gives diagrams of the normal pressures, positive and negative, on three models of roofs making angles of 30 deg., 45 deg., and 60 deg. with the horizontal. In each of these cases the suction on the leeward side of the roof, at right angles to it, equalled one-half the normal pressure on the vertical wall on the windward side. The normal pressures on the windward sides of the models were as follows :—

Roof angle 60 deg., the normal pressure was about two-thirds that on the windward side of the vertical supporting wall.

Roof angle 45 deg., the normal pressure was less than one-half that on the vertical wall.

Roof angle 30 deg., the normal pressure was less than one-sixth that on the vertical wall.

This summary is probably as near an approximation as it is possible to get to the actual distribution of the wind pressure on roofs, until further experimental data on the subject are available. Later results on larger models have somewhat increased these figures. In the case of roofs of large span and considerable length, probably the suction on the leeward side towards the eaves near the middle would not be so great as in the case of a short roof.

It will be seen from the above summary that in the case of the roof model inclined at an angle of 60 deg., the sum of the pressure on the windward side and the suction on the leeward side is greater than the pressure on a surface normal to the wind, which agrees with the fact Prof. W. C. Unwin long since pointed out—namely, that the resultant normal pressure on surfaces inclined at certain angles to the wind is greater than on a surface normal to it.

* *Proceedings Inst. C.E.*, vol. clvi., p. 90.

An iron or steel roof truss is generally fixed at one end, and allowed to slide horizontally at the other end, in order to allow for the effects of expansion and contraction.

In order to find the maximum stresses in the members of a roof truss, the stresses have to be considered, (1) due to the vertical loads, (2) due to the wind blowing from the side on which the support is fixed, (3) due to the wind blowing from the side of the sliding support.

The stresses for each of these three cases can be ascertained most readily by drawing the stress diagrams for the three systems of forces taken separately.

I.—*Stresses due to the Vertical Loads*

The total load for each panel is taken as acting one-half at the panel point at each extremity of it. This load being, of course, its proper proportion of the weight of the truss and of the weight of the covering carried by the trusses. Thus, if the panel lengths are equal the load at each intermediate panel point w will be the total vertical load on a length of the roof equal to the distance apart of the principals, divided by the number of panels in the upper chord, and at the panel points at the supports one-half of this amount. The reaction due to the vertical loads will also be vertical, and if the roof is symmetrical they will, of course, be equal.

Take first the case of an ordinary kingpost roof (Fig. 51).

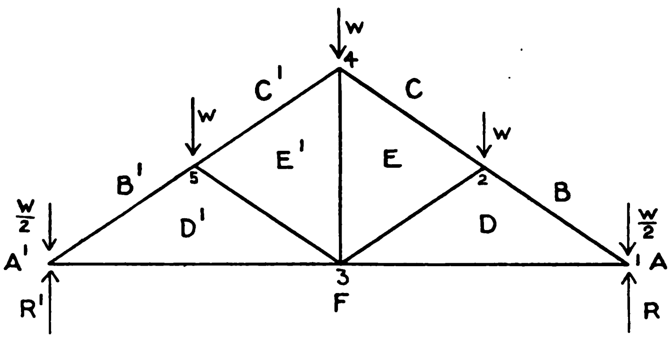


FIG. 51.

Place a letter in the space on each side of the acting loads and reactions, and in each of the triangular spaces between the members. When the roof and loads are symmetrical, the same

Letters may be used on the two sides of the centre line, those on one side being dashed.

In Fig. 51*a* draw the load line $a\,a^1$, using the small letters to mark the points corresponding to the spaces in Fig. 51 represented by the same capital letter. Thus, starting from A and plotting the loads down the line $a\,a^1$, $a^1 f$ will represent the reaction at A^1 and $f\,a$ the reaction at A. We have now to consider the equilibrium of each point in the frame in succession, and in drawing the stress diagram the forces or stresses must be taken in the order in which they act, taken either in the contra-clockwise or clockwise sense. We will here take the former sense.

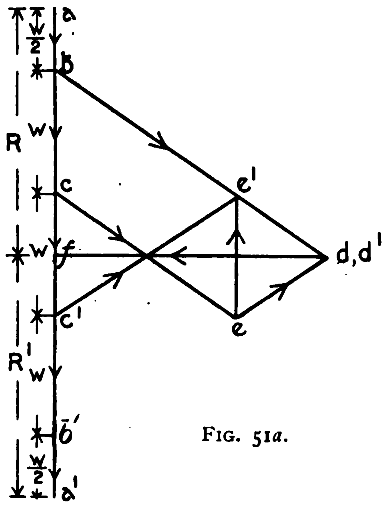


Fig. 51*a*.

Take the panel points in the order as numbered in Fig. 51. The numbering is governed by the condition that as the turn of each panel point comes for consideration, the stresses in all the members meeting at that point, except two, must be known. Starting with the point 1, the forces and stresses acting upon it in order are $f\,a$, the reaction, $a\,b$ equal $\frac{W}{2}$, $b\,d$ the stress in B D, and $d\,f$ the stress in D F, each stress in Fig. 51*a* being of course drawn parallel to its direction in Fig. 51; for the point 2 the force and stresses in order are $d\,b$, $b\,c$, $c\,e$, and $e\,d$; for the point 3 they are $d\,e$, $e\,e^1$, $e^1 d^1$, $d^1 f$, and $f\,d$; and for the point 4 they are $e\,c$, $c\,c^1$, $c^1 e^1$, and $e^1 e$. By following out the above rule as to sense, if the polygon of stresses for any point be considered the direction of the line in the stress polygon, whether away from or towards the point, indicates whether the stress in the corresponding member is a tension or a compression. This is facilitated if an arrow be marked on each line of the stress diagram, giving the direction of the stress along it relative to the first point it applies to.

For ex[illegible] point 1 : $b\,d$ is towards 1, therefore B D

is in compression; again, *d f* is away from 1, therefore A F is in tension.

It will be noticed in Fig. 51*a* that the arrows drawn give

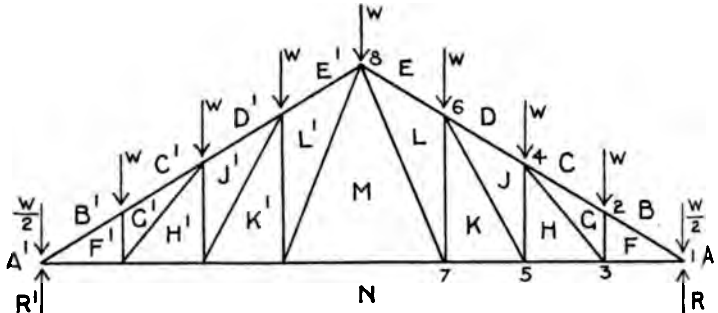


FIG. 52.

the direction of the stress for the first of the two points on which that stress acts. As another example, find whether the stress in E E^1 is tension or compression; consider the stress polygon for the point 3, as this is the first of the points 3 and 4 on which this stress acts, the arrow in Fig. 52 shows that the stress acts away from 3, therefore E E^1 is in tension. In the same way E D is in compression, because the stress *e d* acts towards 2.

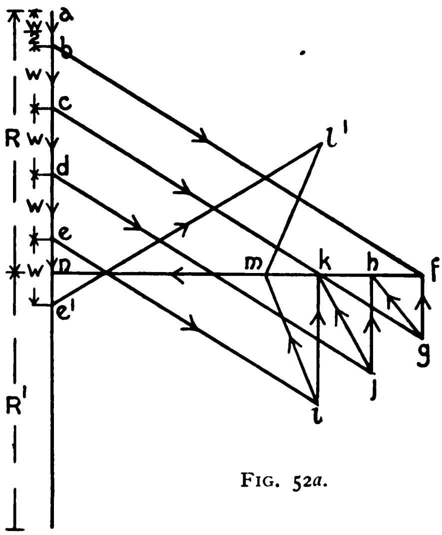


FIG. 52*a*.

Next take the case of a roof in which the upper chords are straight and divided into eight panels with a vertical at each panel point, the foot of each vertical, starting from either point of support, being connected to the top of the next vertical by a sloping brace; the bottom chord being straight as before (Fig. 52), a type suitable for longer spans.

Let the panels be of equal length as before, and call W the panel

load. Plot the loads along the load line downwards, and the reactions upwards (Fig. 52*a*), observing the same rule as to sense as for the forces and stresses acting at one of the panel points. As in the last case, number the joints in Fig. 52 in the order that the stress polygons for the points are drawn in Fig. 52*a*, and on the stress lines as drawn mark the arrow showing the direction of the stress relatively to the first point to which they refer.

For the point 1 the stresses taken in order are *n a*, the reaction at A, *a b* equal $\frac{W}{2}$, *b f* the stress in B F, *f n* the stress in F N; for the point 2 the stresses in the same order are *f b*, *b c*, *c g*, and *g f*; for the point 3 they are *n f*, *f g*, *g h*, and *h n*; for the point 4 they are *h g*, *g c*, *c d*, *d j*, and *j h*; for the point 5, *n h*, *h j*, *j k*, and *k n*; for the point 6, *k j*, *j d*, *d e*, *e l*, and *l k*; for the point 7, *n k*, *k l*, *l m*, and *m n*; and for the point 8, *m l*, *l e*, $e\,e^1$, $e^1\,l^1$, and $l^1\,m$.

The remainder of the diagram is simply a repetition of that for the other half, reversed about the line *n f*.

As before, the arrow in Fig. 52*a* gives the direction of the stress in the corresponding member at the end represented by the lower figure. For example, the member J K is in tension,

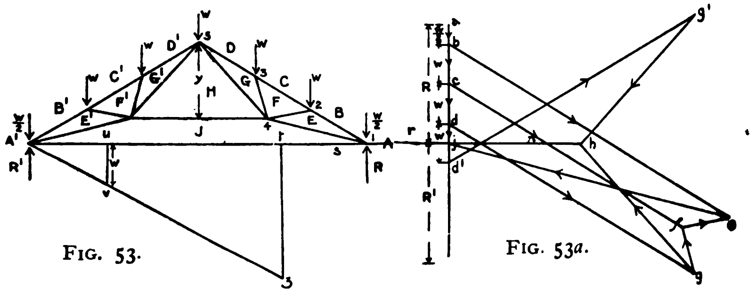


FIG. 53. FIG. 53*a*.

because the stress along it at 5 acts away from 5; the member L K is in compression, because the stress along it at 6 acts towards 6; D J is in compression, because the stress along it at 4 acts towards 4; K N is in tension, because the stress along it at 5 acts away from 5.

Next take the case in which the upper chords are straight and divided into 6 panels, and in which the lower chord is divided into three equal panels, the centre panel being above the level of the supports and the braces being connected to its extremities (Fig. 53).

Plot the loads downwards along the load line (Fig. 53*a*) as before, and the reactions upwards, and number the joints in Fig. 53 in the order they will be taken. It is obvious in this case that the stresses at the joint 4 cannot be determined until those at 2 and 3 are known, otherwise there would be more

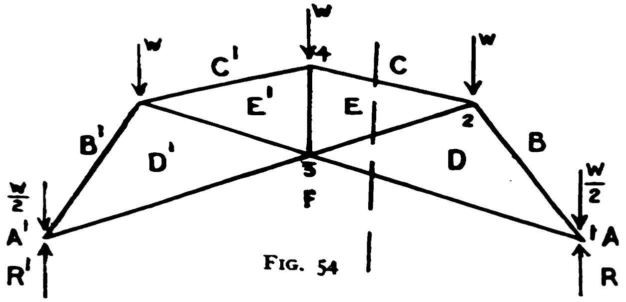


FIG. 54

than two stresses of unknown magnitude, and the polygon could not be drawn. The stresses at 1 are *j a* the reaction at 1, *a b* equal $\frac{W}{2}$, *b e* the stress in B E, and *e j* the stress in E J; at 2 they are *e b*, *b c*, *c f*, and *f e*; at 3, *f c*, *c d*, *d g*, and *g f*; at 4, *j e*, *e f*, *f g*, *g h*, and *h j*; and at 5, *h g*, *g d*, $d\,d^1$, $d^1\,g^1$, and $g^1\,h$. The nature of the stresses can be found as before; for example, the stress in E F is a compression, because the stress along it at 2 acts towards 2; the stress in F G is a compression, because the stress along it at 3 acts towards 3; the stress in G H is a tension, because the stress along it at 4 acts away from 4; the stresses in the upper chords are compressions and in the lower chords tensi[ons].

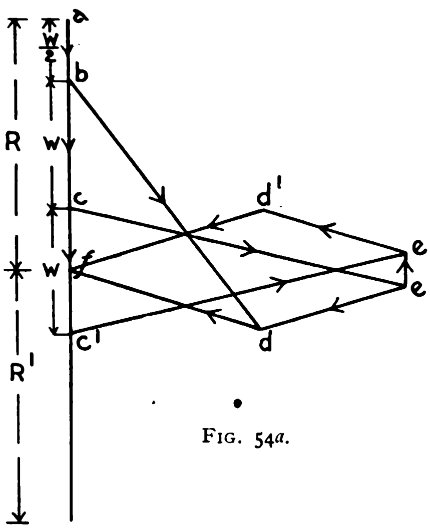


FIG. 54*a*.

It is very desirable in this case to have a check o[n]

position of h in Fig. 53a, since inexactitude in its position may come in owing to lines in Fig. 53a being drawn parallel to the short lines in Fig. 53. When the lower centre member is horizontal, and there is no inclined member attached to it at its centre, this is readily obtained by imagining the roof to be

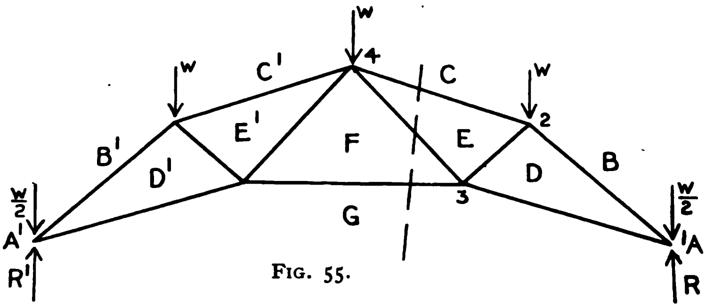


FIG. 55.

severed by a vertical section through the apex and taking moments about that point.

In this case the stress in J H × y equals the resultant upward force at joint 1—*i.e.* $2\frac{1}{2}$ W × half the span minus the load W at joint 2 multiplied by the distance of joint 2 from the vertical through the apex minus the load W at joint 3 × the distance of joint 3 from the vertical through the apex.

Therefore in Fig. 53 make $A^1 r$ equal to two and a half times the half span, $r\ s$ and $s\ t$ equal respectively to the distance of the joints 2 and 3 from the vertical through the apex, and make $A^1 u$ equal to y. Drop perpendiculars and t and on that at u mark off a distance $u\ v$ equal

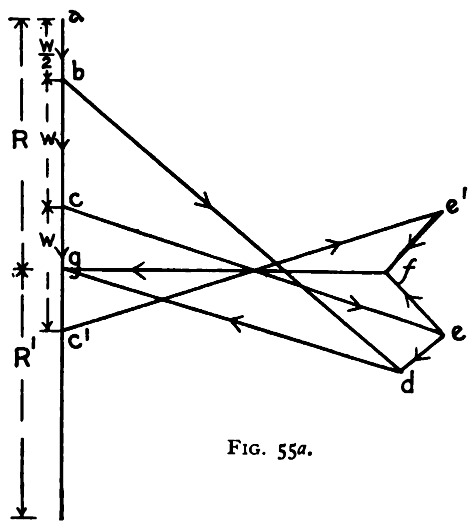


FIG. 55a.

to W, join $A^1 v$ and produce it to z, then $t z$ is obviously the stress in J H, and must therefore be equal to $j h$ in Fig. 53*a*.

Next take the case of a roof with four equal panels in the upper chord, as in Fig. 54, with the ends connected to the opposite second panel point in the top chord and a vertical at the centre.

Draw the loads and reactions along the load line (Fig. 54*a*) as before, and number the joints in Fig. 54 so that in each case the stresses in all but two members meeting at a joint will have been determined before that joint is considered.

At 1 the forces and stresses are $f a$ the reaction, $a b$ equal $\frac{W}{2}$, $b d$ the stress in B D, and $d f$ the stress in D F; at 2 they are $d b$, $b c$, $c e$, and $e d$; at 3, $f d$, $d e$, $e e^1$, $e^1 d^1$, and $d^1 f$; at 4, $e c$, $c c^1$, $c^1 e^1$, and $e^1 e$. The stress in E D is a tension, because the stress along it at 2 acts away from 2. The stress in $E E^1$ is a tension, because the stress along it at 3 acts away from 3. The stresses in the upper chords are compressions and in the lower chords tensions.

Next take the case of four equal panels in the top chord, and three equal panels in the bottom chord, as in Fig. 55.

Draw the load line (Fig. 55*a*), and number the panel points in Fig. 55 so that as the turn of each comes round there will not be more than two of the members in which the stresses are not already determined. The forces and stresses at 1 are, $g a$ the reaction at 1, $a b$ equal $\frac{W}{2}$, $b d$ the stress in B D, and $d g$ the stress in D G; at 2 they are $d b$, $b c$, $c e$, and $e d$; at 3, $g d$, $d e$, $e f$, and $f g$; and at 4, $f e$, $e c$, $c c^1$, $c^1 e^1$, and $e^1 f$.

The stress in E D is a tension, because the stress along it at 2 acts away from 2; the stress in E F is a tension, because the stress in it at 3 acts away from 3. The stresses in the upper chords are compressions and in the lower chords tensions.

A roof truss in which it is impossible so to number the joints as to satisfy the condition that when the turn of any joint comes there shall be not more than two members meeting at it in which the stress has not already been determined, is sometimes called the ambiguous case. To draw the stress diagram it is necessary to introduce another condition. As an example of this we will consider the stresses in the roof truss (Fig. 56), which is made up of two Fink trusses.

It will be observed that when the stresses in the members

meeting at 2 and 3 have been found, at both 4 and 5 there are three members in which the stress is not known. A further condition is therefore necessary, and the simplest method of obtaining one is to suppose the truss divided by a section through the apex, and take moments about the apex of the forces acting on the right-hand half, including the stress in N O at the section

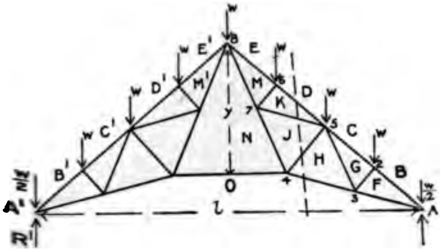


FIG. 56.

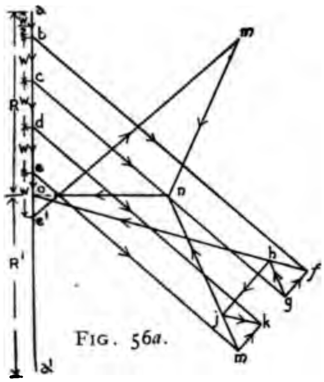


FIG. 56a.

where it is severed; the stresses at the apex will not come in, because moments are taken about that point.

If l is the length of the span, and the panels in the top chords are of equal length, the panel load being W, and if S is the stress in N O, and y the distance of N O from the apex; taking moments about the apex to the right we have, since the resultant of the three loads W at the points 2, 5, and 6 equals 3 W acting at 5—

$$\text{S} \times y = \left(\text{R} - \frac{\text{W}}{2}\right) \frac{l}{2} - 3\,\text{W} \times \frac{l}{4}.$$

Now $\text{R} = 4\,\text{W}$, $\therefore\ \text{S} = \dfrac{\text{W}\,l}{4\,y}(7 - 3) = \dfrac{\text{W}\,l}{y}$.

Thus, if $y = \dfrac{l}{3}$ say, $\text{S} = 3\,\text{W}$, and is obviously a tension. This enables the stress diagram to be drawn.

Plot the load line $a\,a^1$ and reactions $a^1\,o$ and $o\,a$ as before (Fig. 56a). Number the joints as indicated in Fig. 56, the joint 4 should be as there placed, because we have found the stress in N O, and therefore when the turn of that joint comes we know the stresses in all the members meeting there except two, and the same result obtains for the joint 5.

At the joint I the forces and stresses are $o\,a$ the reaction, $a\,b$ equal $\frac{W}{2}$, $b\,f$ the stress in B F, and $f\,o$ the stress in F O; at 2 they are $f\,b$, $b\,c$, $c\,g$, and $g\,f$; at 3, $o\,f$, $f\,g$, $g\,h$, and $h\,o$; at 4, $o\,h$, $h\,j$, $j\,n$, and the predetermined stress $n\,o$; at 5, $j\,h$, $h\,g$, $g\,c$, $c\,d$, $d\,k$, and $k\,j$; at 6, $k\,d$, $d\,e$, $e\,m$, and $m\,k$; at 7, $n\,j$, $j\,k$, $k\,m$, and $m\,n$; and at 8, $n\,m$, $m\,e$, $e\,e^1$, $e^1\,m^1$, and $m^1\,n$. It will be noticed that since J N and M N are in the same straight line, the point j in Fig. 56a must lie on the straight line $m\,n$, which is a check on the work.

If any of the trusses be imagined to be severed by a section the stresses in the members so severed must, of course, be in equilibrium with the external forces acting on the portion of the truss on either side of the dividing section. This being the case, the lines in the stress diagram representing the stresses in the severed members and those representing the external forces acting to either side of the section, must form a closed polygon. For example, consider the section represented by the dotted line in Fig. 54. The members severed by it are C E, E D, and D F, the external forces acting to the right of the section are F A, A B, and B C, and the corresponding lines in the stress diagram obviously constitute the closed polygon $f\,a\,b\,c\,e\,d\,f$, which is also obvious from an inspection of Fig. 54a. The external forces to the left of the section are F A^1, $A^1\,B^1$, $B^1\,C^1$, and C^1 C, and the corresponding lines on the stress diagram obviously constitute the closed polygon $f\,a^1\,b^1\,c^1\,c\,e\,d\,f$. Similarly in Fig. 55 the section indicated by the dotted line severs the members C E, E F, and F G, the stresses in which must balance the external loads acting on either side of the section giving the closed polygons in Fig. 55a, $g\,a\,b\,c\,e\,f\,g$ and $g\,a^1\,b^1\,c^1\,c\,e\,f\,g$ respectively. In Fig. 56 the section indicated by the dotted line severs four members of the truss D K, K J, J H, and H O, the stresses in which must balance the external loads acting on either side of the section giving the closed polygons in Fig. 56a, $o\,a\,b\,c\,d\,k\,j\,h\,o$ and $o\,a^1\,b^1\,c^1\,d^1\,e^1\,e\,d\,k\,j\,h\,o$ respectively.

II.—*Stresses due to Wind Pressure and Wind Suction*

As already indicated, the stresses due to the wind can only be approximated to in the absence of fuller experimental data on the subject. It has also been intimated that the suction on the leeward side is of equal importance with the pressure on

the windward side. The stresses due to the wind have to be ascertained with the wind blowing from either side, because the stresses are different according as the wind blows from

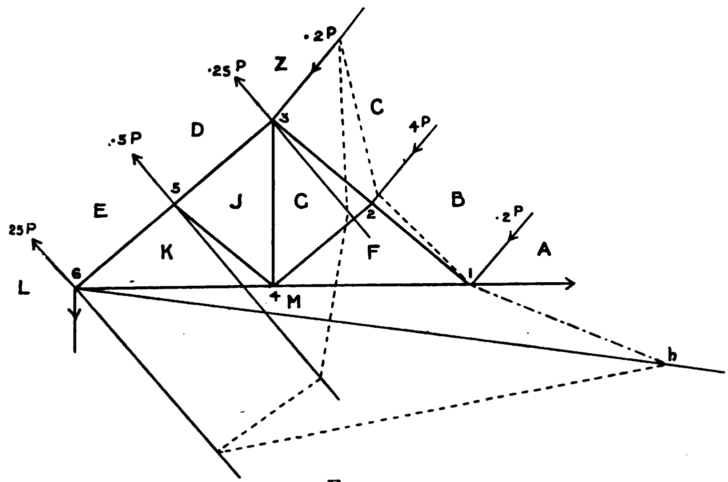


FIG. 57.

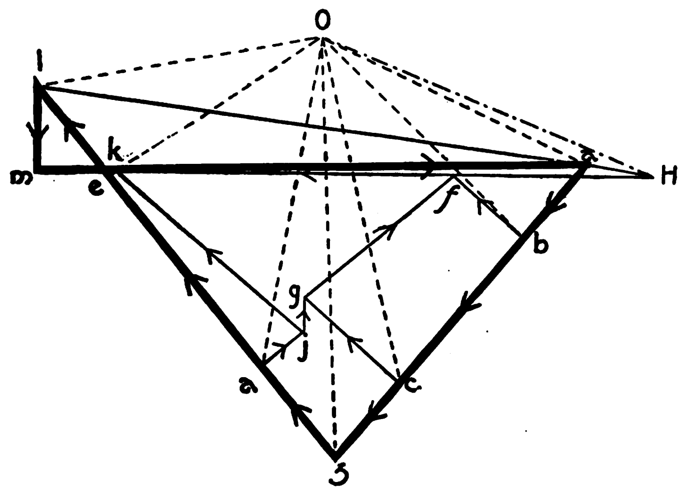


FIG. 57*a*.

the side of the sliding or the fixed support. The method of drawing the stress diagrams due to the wind is illustrated in Figs. 57, 57*a*, and 58, 58*a*, the first two diagrams referring to the

wind, when blowing from the fixed end, 1, and the second two when blowing from the sliding end, 6.

As the angle of the roof is 40 deg. the pressure on the windward side is taken as 0·4 of that on the windward side of a vertical surface; if P be the pressure on such a vertical surface of an area equal to the panel length multiplied by the distance apart of the trusses, at joint 1 and 3 in Fig. 57, the pressure will be 0·2 P, and at joint 2 it will be 0·4 P. The suction on the leeward side is taken equal to 0·5 P, therefore at joints 3 and 6 there is a suction equal to 0·25 P, and at joint 5 a suction equal to 0·5 P. First draw the load diagram (Fig. 57*a*) *a b c z d e l*, *a l* is, of course, the resultant of the acting loads. In order to find the reactions, take any pole O in Fig. 57*a* and join O to each point on the load polygon. Through joint 6 in Fig. 57 draw a line parallel to the resultant *a l* (Fig. 57*a*). Starting at 1, draw a line parallel to O *b* to meet the load B C acting at 2, from that point of intersection draw a line parallel to O *c* to meet the load C Z acting at 3; from this point draw a line parallel to O *z* to meet the load Z D acting at 3; from the latter point of intersection draw a line parallel to O *d* to meet the load D E acting at 5; from this point draw a line parallel to O *e* to meet E L, the load at 6; and from that point draw a line parallel to O *l* to meet the line through 6 drawn parallel to the resultant, in *h*; then 1 *h* is the closing line of the equilibrium polygon. From O draw a line O H parallel to the closing line 1 *h* to meet *l a* produced in H. It will be noticed that the point H falls beyond *a* from *l*, therefore the components of the reactions at 6 and 1 parallel to the resultant are *l* H and H *a*. From H draw a horizontal line H *m*, parallel to the closing line $\overline{16}$ of the true equilibrium polygon. Since the reaction at the joint 6 is vertical, *l* H and H *m* are its components along the resultant and along the true closing line, therefore this reaction is represented by *l m* and acts downwards; in other words, there is an uplift at 6 due to the wind which will be counteracted by the weight. The components of the reaction at 1 are *m* H parallel to the closing line, and H *a* parallel to the resultant, therefore *m a* is this reaction. The force polygon is therefore *a b c z d e l m a*. The stress diagram due to the wind blowing from the "fixed" side can now be drawn. At 1 the forces and stresses acting are *m a* the reaction, *a b* the load, *b f* the stress in B F, and *f m* the stress in F M; at 2 they are *f b*, *b c*, *c g*, and *g f*; at 3, *g c*, *c z*, *z d*, *d j*, and *j g*; at 4, *m f*, *f g*, *g j*, *j k*, and *k m*; at 5, *k j*, *j d*, *d e*, and *e k*. It will be noticed

that the fact that *j k* and *e k* intersect on the line *m* H is a check on the accuracy of the drawing; at 6 the forces and stresses are *m k*, *k e*, *e l*, and *l m*. It will be observed that B F and C G

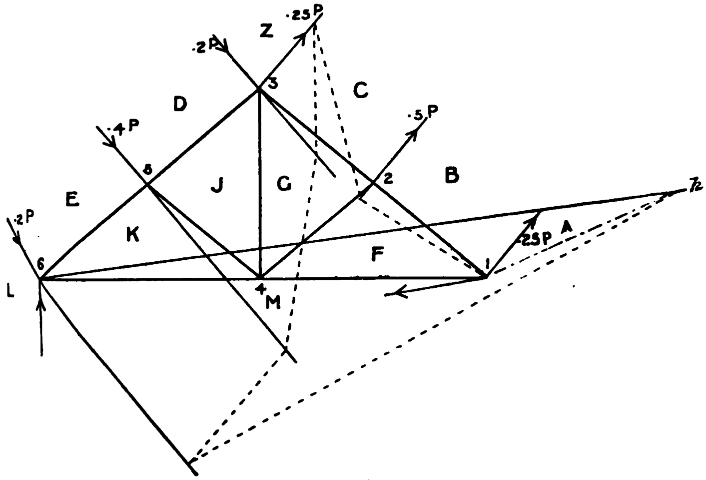


FIG. 58.

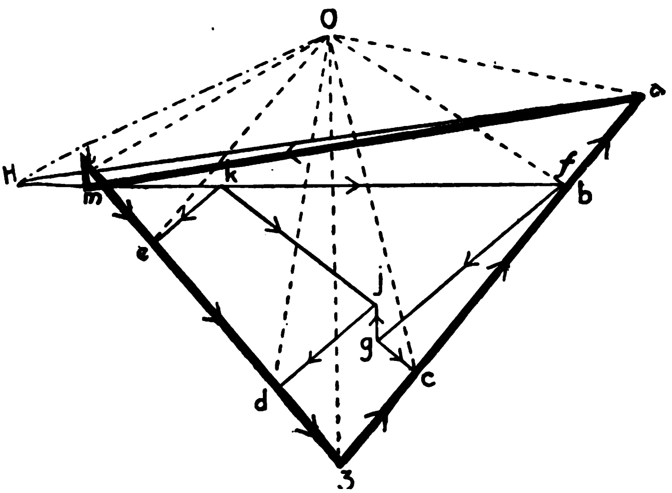


FIG. 58*a*.

are in tension; D J and E K are in compression; F M and K M are in tension; F G and G J are in compression, and J K is in tension.

Next take the case of the wind blowing from the left, or the "sliding" side. The loads in Fig. 58 will change sides compared with those in Fig. 57. Draw the load diagram in Fig. 58*a*. It is represented by *a b c z d e l*, and *l a* is the resultant of the loads. Draw a line through 6 parallel to this resultant. In order to find the components of the reactions parallel to it, take any convenient pole O in Fig. 58*a* and join O to the extremities of the lines representing the loads. Starting at 1, draw an equilibrium polygon exactly as before; the last line meets the line through 6 parallel to the resultant in *h*, and 1 *h* is the closing line of the equilibrium polygon in question. Therefore from O draw O H parallel to 1 *h* to meet *a l* produced. The components of the reactions at 1 and 6 parallel to the resultant are *a* H and H *l*. Through H draw a line parallel to the closing line $\overline{1\,6}$ of the true equilibrium polygon. Since the reaction at the joint 6 is vertical *m* H and H *l* are its components parallel to the closing line and resultant respectively, and *m l* is its amount —in this case it acts upwards—and the reaction at 1 is *a m* with components *a* H and H *m*, parallel to the resultant and closing line respectively; *a m* acts in the downward direction, so that there is an uplift at 1 due to the wind. The polygon of forces therefore is *a b c z d e l m a*. The stress diagram for the wind blowing from the "sliding" side can now be drawn. The forces and stresses acting at 1 are, taking them in the clockwise sense to suit their direction, *b a* the load at 1, *a m* the reaction at 1, *m f* the stress in M F, and *f b* the stress in F B; at 2 they are *c b, b f, f g*, and *g c*; at 3, *d z, z c, c g, g j*, and *j d*; at 4, *j g, g f, f m, m k*, and *k j*; at 5, *e d, d j, j k*, and *k e*. Here again the accuracy of the work is tested by *j k* and *e k* intersecting on the line H *f*; at 6 the forces and stresses are *l e, e k, k m*, and *m l*. In this case B F and C G are in compression; D J and E K are in tension; F M and K M are in compression; G F is in tension; G J and J K are in compression. It will be observed that the sign of the stresses in all the members is changed except that in G J.

The method of proceeding for other shapes of roofs is similar, the only uncertain point being the distribution of the wind loads, which is, of course, the primary data for the problem.

To determine the resultant stress in any member of the roof truss, to the stress in it due to the vertical load must be added the maximum stress *of the same sign* due to the wind blowing from either side; it is also necessary to find the difference

between the stress due to the vertical load and the maximum stress of the opposite sign due to the wind. These two results give the variation of the stress in the member. If the stress in the member due to the wind from both sides is of the same sign but of the opposite sign to the stress in the member due to the vertical load, the latter stress will be one limit of the variation, and the difference between it and the maximum wind stress the other limit of variation.

CHAPTER V

CROSS-SECTIONAL AREA OF THE MEMBERS OF A STRUCTURE

IN this chapter it is proposed to consider how to obtain the cross-sectional areas of the members when the maximum stresses and the variation of the stresses they have to sustain have been calculated.

In the first place, it may be stated that the permissible intensity of working stress or the unit working stress is taken either directly or indirectly as the breaking intensity of stress divided by a factor of safety. In arriving at the value of this factor of safety the first consideration is that it is absolutely necessary that the highest intensity of stress that any member of a structure, intended to be permanent, is to be subjected to from time to time shall be less than the elastic limit, otherwise that member would be permanently deformed by the stress. The elastic limit is by no means a definite proportion of the breaking stress, but it appears to have a "natural" value which may be found after a specimen has been subjected several times to a stress less than the elastic limit. If the stress temporarily exceeds the elastic limit, the effect is—at all events temporarily—to raise that limit. For mild steel the elastic limit is about one-half the breaking intensity of stress, which makes a factor of safety of at least 2 absolutely essential. But in addition to this, small imperceptible flaws and want of homogeneity in the material, the fact that the centre of stress on the section of a member very often does not coincide with its centre of figure—thereby producing a non-uniform intensity of stress on the section—and the possibility of imperfect workmanship, have to be provided for. The maximum intensity of stress in a section is also affected by what is known as "impact," or by the application of the moving loads or parts of them with more or less shock, as distinguished from a gradual application of the load. The above are effectively provided for, as will be further explained in what follows, if the effect of what is known as "fatigue" of the material is counteracted by a sufficient reduction of the unit working

stress. It is well known that if a load be applied suddenly to an elastic member, the latter will be deformed twice as much as if the load were applied gradually, because the work done during deformation by the internal stresses must equal the work done by the suddenly applied load. Now the stress to begin with is zero, and it increases proportionally to the extension or compression, as the case may be, up to a maximum; its average value is therefore one-half the maximum, and this multiplied by the deformation equals the load multiplied by the same distance. Therefore the maximum intensity of stress momentarily produced is twice the intensity of the load.

Suppose there is present in the member to begin with, say, a pull P, and a thrust P^1, say, is suddenly superimposed, Fig. 59, the external work done $= P - P^1 \times$ the deformation, A B; since the work done by the internal stresses must be equal to this, the area shaded horizontally in Fig. 59 must be equal to the area shaded vertically, therefore C must be the centre of A B and the final stress $= P^1 - P + P^1 = 2P^1 - P$, and the stress varies from $-P$ to this value; thus it increases by twice the amount it would do if the load P^1 were applied gradually; in other words, the maximum stress instead of being $P^1 - P$ is $2P^1 - P$, or the static stress due to the maximum load + the variation of stress. In the case of the members of such a structure as a bridge, the stress, as measured by the deflection, is somewhat increased above that due to the load applied gradually, but it does not actually in most cases reach anything like double the amount, even when it is borne in mind that it is not only the sudden application of the load as a whole which has to be considered, but the suddenness of application of any parts of the load—*e.g.* local vibrations and therefore increments of stress, which are caused by lack of perfect balance in the locomotive driving-wheels, by lurching of the locomotive or waggons from side to side due to temporary inequalities in the track or other reasons—

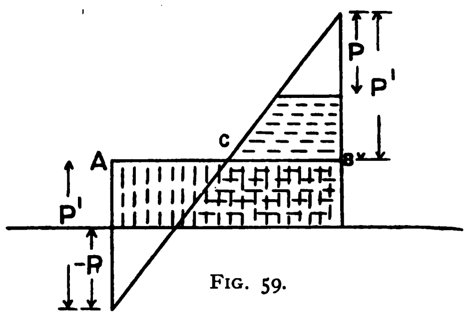


FIG. 59.

especially if the period of such oscillations happens to coincide with the natural time of vibration of the bridge. Rail joints on a bridge are a fruitful source of vibration, but are avoided as far as possible, particularly towards the centre of a bridge. It will be obvious that any lack of uniformity in the distribution of stress in the section of a member naturally increases the maximum intensity of stress developed due to the above causes.

The experiments of Wöhler, Spangenberg, Baker, Bauschinger, Stanton, and others, have well established the facts that if a member is subjected to a load varying from zero to a maximum, it will in course of time break down under a load whose maximum value equals about half the static breaking load; and if the variation is from a pull to a thrust of the same intensity, that a pull or thrust which is a much smaller proportion of the static breaking weight will ultimately cause the rupture of the specimen.

These results are generally ascribed to "fatigue," but why the material should suddenly break off quite short after the application of a load which never exceeds one-half the breaking load of the particular member is not generally satisfactorily explained. The effects of impact and fatigue are bound up with each other, because when the live load is applied with a certain amount of suddenness the stresses induced are, as just explained, greater than the stresses due to the same load quietly applied; and the fatigue depends principally upon the relative value of the maximum stress, produced by the more or less suddenly applied loads, and the elastic limit of the material. The following considerations, although perhaps not a complete explanation, remove from the mind any difficulty in realising that it is quite to be expected that if the impact or lack of uniformity in the distribution of stress on a section, due to a load, causes the maximum average stress in a member to approach its elastic limit, that the member will in course of time probably be ruptured, under the repeated application of the load.

If it be admitted that the material of which a member is made is not so homogeneous that, even if the load is applied perfectly axially, the stress will be uniform all over the section, but that it will exceed the average at some parts of the section and be less than the average at other parts of the section, it will follow that if the load exceeds a certain limit—approaching the elastic limit, it is quite possible that in a certain cross-section, for small portions of the area, the elastic limit of the

material may be exceeded, and those particles would stretch and become harder than the rest of the section. This would make the section rather less homogeneous than before, and the next time the load was applied the elastic limit of stress on certain other particles might be exceeded, with the result that these particles would stretch and become harder. Thus the section does not tend to become more homogeneous, but the reverse, as different particles in it are from time to time stretched beyond their elastic limit.

Now if a specimen be placed in a testing machine and stretched up to the yield point, and the load then taken off, it will be found that the next time the load is applied the yield point has been raised; and if each time the load is applied it is allowed to just exceed the yield point, and is then immediately taken off, the yield point continues to rise, until after a number of applications and removals of the load the specimen will finally break without the yield point having been exceeded, the load being considerably less than if the specimen had been tested to destruction in the first instance. Applying this result to the case of individual particles in the cross-section of the member under consideration, it will be seen that in the course of innumerable applications of the live load the same particle may from time to time be again and again stretched beyond its varying elastic limit, particularly as the material in the cross-section will tend to get less homogeneous as more particles in it are stretched beyond their elastic limits; and, after perhaps millions of applications, one group of particles may snap for the same reason as the specimen in the testing machine snapped. When this has occurred for one group of particles it is likely to occur for others, so that it would only be a matter of time for a complete fracture of the member to occur at the section in question under a load which is considerably less than the static breaking weight. Since this action takes place in the particles of the material, it is no matter for surprise that there is no external evidence of alteration of sectional area before fracture, for in the case, for instance, of testing a round bar to destruction by torsion, when the action is to stretch individual particles, there is no noticeable reduction in external section, but individual fibres become denser and farther apart.

Consequently, to prevent a member breaking with repetition of loading, it is necessary that the load should be small enough, taking into account the want of homogeneity of the mater

to prevent the elastic limit being exceeded at any point in a section; if this condition is complied with, experience indicates that the instantaneous increases of stress due to impact are in most cases amply provided for, and the life of the member will not be endangered. It would thus appear that the increase of the stress above the elastic limit due to want of perfect homogeneity, when the average stress applied is less than the elastic limit of the material, is a possible explanation of the cause of fatigue.

There are a number of methods in common use to limit the working intensity to a safe amount; three of these are given in what follows, and although they approach the question from different standpoints, the general result is much the same, in that the members most affected by the variation of stress due to the live load are subjected to less unit working stress than those members less interfered with by it. Fatigue is thus avoided and the effects of impact provided for.

(1) From considerations similar to those on page 103, Prof. T. Claxton Fidler has proposed the formula for the area A:

$$\text{A} = \frac{\text{S}}{\frac{p}{r}} = \frac{\text{max. S} + w}{\frac{p}{r}}.$$

Max. S is the maximum stress due to dead and live load as calculated, S is the "increased" stress, *i.e.* max. S increased to allow for the more severe effect of the live load, p is the intensity of breaking stress of the material, measured, as is usual, relatively to the initial area of the member, r is the factor of safety if all the load were dead load, *i.e.* when S = max. S, and w is one-half the total variation of stress for the flanges where the variation of stress is fairly gradual, and for other members of a bridge it equals the total variation of stress. Although the method of derivation of this formula is open to criticism, it has the desired effect of limiting the unit working stress in a member according to the variation of stress in it. For comparison with the formulas which follow these may be written—

For *flanges*—

$$\text{A} = \frac{\text{max. S} + \frac{\text{max. S} - \text{min. S}}{2}}{\frac{p}{r}}$$

$$= \frac{3\ \text{max. S} - \text{min. S}}{\frac{2p}{r}}.$$

For *other members*—

$$\text{A} = \frac{\text{max. s} + \text{max. s} - \text{min. s}}{\frac{p}{r}}$$

$$= \frac{2\ \text{max. s} - \text{min. s}}{\frac{p}{r}}.$$

Min. s is the minimum stress in the member due to dead and live load together, in the flanges it equals the dead load stress, but in the bracing members it may be less than the dead load stress or of the opposite sign to it.

(2) The formula known as the Launhardt-Weyrauch formula, intended to conform with the results of Wöhler's experiments, gives an expression for the permissible working intensity of stress, and one form of it is—

$$\text{Intensity of working stress} = \frac{p}{\rho} = \frac{2}{3r}\, p \left(1 + \frac{1}{2}\,\frac{\text{min. s}}{\text{max. s}}\right).$$

Where ρ is the "true" factor of safety, and the other symbols have the same meaning as before, the sign between the two terms will of course be $+$ when min. s is the same sign as max. s, and $-$ where they are of opposite sign.

It may be written for comparison—

$$\text{A} = \frac{\text{max. s}}{\text{working intensity of stress}}$$

$$= \frac{3\,r\,(\text{max. s})^2}{p\,(2\ \text{max. s} + \text{min. s})}.$$

(3) Another method, which is simpler in application, consists in multiplying the stress due to moving load by a certain coefficient, 1·5 for the flanges of a girder and 2 for the other members, adding this result to the dead load stress and dividing the sum by a constant working intensity of stress (say $\frac{p}{r}$ as in the first case), in order to find the area.

The formula for *flanges* is therefore—

$$\text{A} = \frac{\text{s}}{\frac{p}{r}} = \frac{\text{max. s} + \frac{\text{max. s}_{\text{L}}}{2}}{\frac{p}{r}}$$

where S is now the "increased" stress by this method, max. S_L is the maximum stress due to live load and equals max. S − min. S for flanges, because the maximum stress in the flange is that due to dead load and live load, and the minimum stress is that due to dead load alone. Therefore for the flanges $A = \dfrac{3 \text{ max. S} - \text{min. S}}{2\dfrac{p}{r}}$, which is the same formula as in method (1).

The formula for other members is—

For *other members*—

$$A = \frac{S}{\frac{p}{r}} = \frac{\text{max. S} + \text{max. S}_L}{\frac{p}{r}}.$$

In the case of web members max. S_L is less than the variation of stress in them, because the latter = max. S_L − min. S_L, and min. S_L is of the opposite sign in this case to max. S_L, therefore (1) gives a larger area for web members than this.

If max. S = min. S—*i.e.* the stress is constant and there is no live load—all the formulas give the same result.

If min. S = − max. S, the second formula in (1) gives the same result as the formula in (2), but the second formula in (3) gives a different value. However, this case does not occur in bridge work.

Writing $A = \dfrac{S\,r}{p} = \dfrac{\text{max. S} \times \rho}{p}$, where S, the "increased" stress in the member, and the other symbols are as before; it will be noticed that methods (1) and (3) increase the value of max. S to S and leave r unaltered, to regulate the area of a member according to the range of stress in it; and the method (2) increases the value of r to ρ, for the same purpose, and leaves max. S unaltered. It is obvious that increasing either max. S or r has the effect of increasing the cross-sectional area, and therefore of reducing the working intensity of stress.

The third method has the advantage of being easily applied, and it is probably as accurate as any. To apply it, it is only necessary to add to the calculated maximum stresses one-half the live load stresses for the flanges and the total live load

stresses for the other members, and divide the result by $\frac{p}{r}$, which for mild steel, if $r = 3$, may be taken as 9 tons per square inch for tension, 7 tons per square inch for compression, 6 tons per square inch for shearing in rivets, and 12 tons per square inch for bearing pressure in rivets. When there is any doubt that each component part of a member will take its proper proportion of the stress, r must be taken as greater than 3.

It will be observed that these intensities simply represent what the unit working stress would be if the load were constant, but as part of the load is a live load, the actual unit working stress is less—*e.g.* suppose the stress Q in a member of the bracing due to live load is equal to twice the stress P, due to dead load; then

$$\text{A} = \frac{\text{P} + 2\text{Q}}{9} = \frac{\text{P} + \text{Q}}{9 \times \dfrac{\text{P} + \text{Q}}{\text{P} + 2\text{Q}}} = \frac{\text{P} + \text{Q}}{9 \times \frac{3}{5}} = \frac{\text{P} + \text{Q}}{5{\cdot}4}.$$

Thus, for this ratio of live to dead load, the unit working intensity of stress = 5·4 tons per square inch. Similarly for shearing and bearing stresses, and for any other ratio of Q to P.

In American practice the fraction of the live load stress added is sometimes made to depend on the length of the portion of the span covered by the load which causes the maximum stress in any given member. Mr. J. A. L. Waddell * gives $\frac{400}{\text{L} + 500}$ as the fraction of the equivalent live load to be added for impact in the case of railway bridges; L being the length in feet of the portion of the span covered by the live load which causes the maximum stress in the member under consideration. Sometimes a similar fraction is applied to the range of stress in the member, the range of stress being, of course, greater than the maximum stress in those members in which different positions of the live load cause stresses of opposite sign.

In the case of members in compression it is necessary to reduce the working intensity of stress still further as the length of the member increases and as the stiffness of the cross-section in any direction decreases, in order to provide against the

* *D* York, J. A. L. Waddell, p. 152.

tendency to buckle. The formula generally made use of is Prof. Rankine's modification of Gordon's formula for long columns. This formula is—

The intensity of working stress

$$= \frac{\text{max. S}}{\text{A}} = \frac{\text{S}}{\text{A}} \times \frac{r}{\rho} = \frac{1}{\rho} \times \frac{f}{1 + c\frac{\text{L}^2}{k^2}},$$

where L is the length and k the radius of gyration in the same units, k being taken with reference to the axis of the cross-section about which the radius of gyration is least; f and c are constants, their values being given in the following table:—

Material.	Ends fixed.		Ends hinged.	
	f	c	f	c
Mild Steel	21	$\frac{1}{30000}$	21	$\frac{1}{7500}$
Hard Steel . . .	31	$\frac{1}{20000}$	31	$\frac{1}{5000}$
Wrought Iron . . .	16	$\frac{1}{36000}$	16	$\frac{1}{9000}$
Cast Iron (hollow or solid cylinders) . . .	36	$\frac{1}{6400}$	—	—

It will be noticed from the table that fixing the ends instead of hingeing them is equivalent to halving the length of the strut.

The formula for struts or columns may therefore be written—

$$\frac{\text{max. S}}{\text{A}} = \frac{1}{\rho} \times \frac{f}{1 + c\frac{\text{L}^2}{k^2}} \quad \text{or} \quad \frac{\text{S}}{\text{A}} = \frac{1}{r} \times \frac{f}{1 + c\frac{\text{L}^2}{k^2}}$$

A trial value for A has to be taken, noting that $\frac{\text{S}}{\text{A}}$ must be less than $\frac{f}{r}$; if the term on the right works out to be less than $\frac{\text{S}}{\text{A}}$ with the tentative value of A taken, then A or the minimum value of k^2 for the proposed cross-section, or both,

must be increased and further trials made until an equality is obtained.

It will be observed that for short struts when L is small, the second term in the denominator of Rankine's formula, which involves L^2, will be small, and the formula does not reduce the working intensity of stress on account of the member in question being a strut; but if L is great—particularly if k is small—the second term assumes considerable importance, increasing the denominator and therefore reducing the working intensity of stress.

Rankine's formula thus gives intermediate values for the working intensity of stress in a column between the two limits:—when L is small and the second term in the denominator disappears—*i.e.* when the effect of buckling is negligible—and when L is great when the second term in the denominator is great compared with the first term unity. Thus the upper limit of the formula would be when unity in the denominator can be neglected in comparison with the second term, then the working intensity of stress $= a \dfrac{k^2}{L^2}$ where a is a constant. This agrees with Euler's theoretical formula for long columns—viz. the working intensity of stress $= \dfrac{4\pi^2 E}{r} \times \dfrac{k^2}{L^2}$

CHAPTER VI

STRESSES ON PLANES THROUGH A POINT IN A MEMBER

THE expression "intensity of stress at a point in a member" is meaningless, unless it is understood on which plane passing through the point in question the stress is acting—except for liquids when the pressure stress is the same on all planes through the point. In the case of other bodies the intensity of stress at a point generally varies on the different planes through the point at which the stress is considered, and it will be shortly proved that, generally speaking, there is one plane through the point on which the intensity of stress is a maximum, and on the plane at right angles to this there acts either the minimum stress of the same sign or the maximum stress of the opposite sign at the point. A clear conception of the variation of the stress on different planes through a point is of vital importance in many parts of the subject, such as the consideration of the maximum intensities of stresses in beams, retaining walls, dams, and in connection with earth pressures; whereas a lack of appreciation of it is apt to lead to structures being designed in which the stresses on certain planes are greater than the maximum working stresses it was intended to allow, or in which stresses of a nature not anticipated are present.

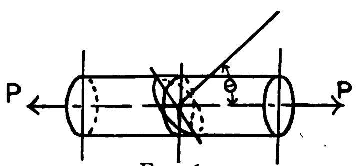


FIG. 60.

In most cases there is a plane normal to which the stresses may be neglected, which is a plane of symmetry with respect to the stresses, and it is only necessary to consider stresses acting at the point on planes at right angles to this plane. Internal forces in a structure are generally referred to as stresses, and the stress per unit of area of a plane as the intensity of stress on that plane at a point. If a single force P (Fig. 60) is acting on a straight round rod of cross-section A, in the direction of

its axis, the intensity of stress on any cross-section at right angles to its axis equals $\frac{P}{A}$. On a plane inclined at an angle θ to this cross-section, the intensity of stress would be $\frac{P}{A \sec \theta} = \frac{P}{A} \cos \theta$, because the area of section of the rod in this plane equals A sec θ. The total stress P will not act normal to this new section, but will be inclined to its normal at an angle θ, and may therefore be resolved normally and tangentially to this plane, the former component is the normal component of stress on the plane, and the latter is the shearing stress in the plane.

The normal component of the intensity of stress

$$= \frac{P}{A} \cos \theta \times \cos \theta = \frac{P}{A} \cos^2\theta,$$

and the shearing stress in the plane

$$= \frac{P}{A} \cos \theta \times \sin \theta = \frac{P \sin 2\theta}{2A}.$$

It is therefore obvious that the normal component of stress is a maximum when $\cos^2\theta = 1$, *i.e.* when $\theta = 0$; and the shearing stress is a maximum when $\sin 2\theta = 1$, *i.e.* $2\theta = 90$ deg. or $\theta = 45$ deg.

If we now consider a cube (Fig. 61) acted upon by the force P as before, and in addition by a force Q acting at right angles to P, P and Q both acting along axes of the cube, which may be taken as the axis of x and the axis of y respectively. Then the plane containing P and Q will be the plane normal to which the stresses may be neglected. Consider the plane as before, whose normal is inclined to P at an angle θ; its normal will be inclined to Q at an angle $3\frac{\pi}{2} + \theta$ or $-\left(\frac{\pi}{2} - \theta\right)$; therefore the component of Q normal to the plane

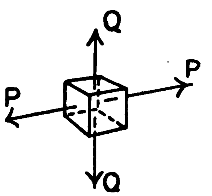


FIG. 61.

$$= \frac{Q}{A} \cos\left(\frac{\pi}{2} - \theta\right) \times \cos\left(\frac{\pi}{2} - \theta\right) = \frac{Q}{A} \sin^2\theta,$$

and the shearing stress in the plane due to Q

$$= -\frac{Q}{A} \cos\left(\frac{\pi}{2} - \theta\right) \sin\left(\frac{\pi}{2} - \theta\right) = -\frac{Q \sin 2\theta}{2A}.$$

The total normal component of stress on this plane, whose

normal is inclined at an angle θ to the axis of x, therefore

$$= \frac{P}{A} \cos^2\theta + \frac{Q}{A} \sin^2\theta,$$

and the total shearing stress

$$= \left(\frac{P}{A} - \frac{Q}{A}\right) \times \frac{\sin 2\theta}{2},$$

or if we write $\frac{P}{A} = p_x$ and $\frac{Q}{A} = p_y$, p_x being taken as the greater of the two stresses, the normal component of stress on the plane

$$= p_x \cos^2\theta + p_y \sin^2\theta,$$

and the shearing stress

$$= \frac{p_x - p_y}{2} \times \sin 2\theta.$$

It is thus obvious, therefore, that on any other plane except those normal to the axes of x and y the stress has both a normal and tangential component; in other words, the resultant stress on the plane is inclined to it at an angle other than a right angle. Next, to prove that when *the resultant stresses on any two planes at right angles are known, there are always two planes at right angles to each other to which their resultant stresses are normal.* The planes to which the resultant stress is normal are called planes of "principal" stress. The two planes normal to P and Q through the centre of the cube in Fig. 61 are such planes. If we consider the stresses acting on a small cube (Fig. 62) of the member under consideration, since the cube is at rest the stresses normal to A B, B C, C D, and D A must balance. As the cube is supposed to be infinitely small, the shearing stress on C D is equal and opposite to that on A B; similarly the shearing stress on D A is equal and opposite to that on B C; because the shear on A B is that exerted by part of the body below A B on A B, and the shear on C D is the equal and opposite shear exerted by the portion of the body above C D on C D, similarly for B C and D A, and the moment of the tangential stresses on A B and C D must balance the moment of the tangential stresses on B C and D A; therefore the shear on A B is equal to that on B C. In other words, the intensity of the shearing stress on any two planes at right angles must be equal. This is also obvious from the formula just deduced for the shearing stress on a plane whose

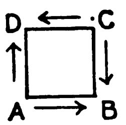


FIG. 62.

normal is inclined at an angle θ to the axis of x—viz. the shearing stress equals $\dfrac{p_x - p_y}{2} \times \sin 2\theta$, which expression has the same numerical value if $\theta = \dfrac{\pi}{2} + \theta$; the sign, however, is different, which means that the part of the body below A B exerts on A B a shear of the opposite sign to what the cube exerts on A D. Therefore the shear exerted by the portion of the body to the left of A D on A D, which is equal and opposite to the latter, will be in the same direction as the shear on A B—*i.e.* away from A.

Suppose, therefore, that at O (Fig. 63) the normal and tangential components of the intensity of stress on the plane O A are p_n and p_t, and on the plane O B at right angles to O A these components are p_n^1 and p_t, and consider the equilibrium of the triangle O A B, where A B is the plane on which the resultant stress acts normally (if such a plane exists). Call the normal components of the stresses positive when they act towards the inside of the triangle—*i.e.* when they are compressions.

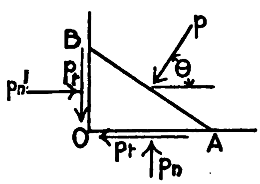


FIG. 63.

Resolving the stresses acting on the triangle horizontally, we have

$$\text{OB} \times p_n^1 - \text{OA} \times p_t = p \times \text{AB} \times \cos\theta,$$

and resolving the stresses vertically we have

$$\text{OA} \times p_n - \text{OB} \times p_t = p \times \text{AB} \times \sin\theta.$$

Substituting $\quad \text{OA} = \text{AB}\sin\theta$ and $\text{OB} = \text{AB}\cos\theta$

we have $\quad p_n^1 \cos\theta - p_t \sin\theta = p\cos\theta \quad . \quad . \quad (1)$

and $\quad p_n \sin\theta - p_t \cos\theta = p\sin\theta \quad . \quad . \quad (2)$

From (1) $\dfrac{p_n^1 - p}{p_t} = \tan\theta$, and from (2) $\dfrac{p_n - p}{p_t} = \cot\theta$.

Multiplying these two results

$$(p_n^1 - p)(p_n - p) = p_t^2,$$

or

$$p^2 - p(p_n + p_n^1) + p_n p_n^1 - p_t^2 = 0.$$

This is a quadratic equation to find p, therefore there are two values, the two roots p_x and p_y of this equation, if real, which would be the resultant stresses which are normal to their planes of action—*i.e.* the two "principal" stresses.

Solving

$$p_x, p_y = \frac{p_n + p_n^1 \pm \sqrt{(p_n + p_n^1)^2 - 4(p_n p_n^1 - p_t^2)}}{2} \quad (3)$$

which may be written

$$p_x, p_y = \frac{p_n + p_n^1 \pm \sqrt{(p_n - p_n^1)^2 + 4 p_t^2}}{2} \quad . \ . \ (4)$$

To find the direction of the planes of principal stress, eliminate p from the equations (1) and (2) by multiplying (1) by $\sin\theta$. and (2) by $\cos\theta$, and subtracting when we get

$$p_t (\cos^2\theta - \sin^2\theta) = (p_n - p_n^1) \sin\theta \cos\theta,$$

or $$\tan 2\theta = \frac{2 p_t}{p_n - p_n^1}$$

$$\therefore \quad 2\theta = \tan^{-1}\frac{2 p_t}{p_n - p_n^1}, \text{ or } \pi + \tan^{-1}\frac{2 p_t}{p_n - p_n^1},$$

or $$\theta = \tfrac{1}{2}\tan^{-1}\frac{2 p_t}{p_n - p_n^1}, \text{ or } \frac{\pi}{2} + \tfrac{1}{2}\tan^{-1}\frac{2 p_t}{p_n - p_n^1}.$$

It therefore is obvious that the planes of principal stress are at right angles to each other, and their direction may be found either from the above formula or from the above equations,

$$\cot\theta = \frac{p_n - p}{p_t} \text{ or } \tan\theta = \frac{p_n^1 - p}{p_t},$$

where $p = p_x$ or p_y, according as the angle θ is being found for the one or the other.

From equation (4) it is obvious that $p_x + p_y = p_n + p_n^1$—*i.e.* the sum of the normal components of stress on any two planes at right angles equals the sum of the principal stresses.

It also follows from the same equation that $p_x - p_y = \sqrt{(p_n - p_n^1)^2 + 4 p_t^2}$. Again, from equation (3) it is obvious that if the second term under the square root—viz., $p_n p_n^1 - p_t^2$—is positive, *i.e.* if $p_n p_n^1 > p_t^2$, the term under the square root is always less than $p_n + p_n^1$, the term outside the square root, therefore both roots are of the same sign, or, in other words, both principal stresses are either compressions or tensions.

If the normal component of stress on one of the planes is zero—*e.g.*, if

$$p_n = o \text{ then } p_x, p_y = \frac{p_n^1 \pm \sqrt{(p_n^1)^2 + 4 p_t^2}}{2}, \text{ and } \cot\theta = -\frac{p}{p_t}, \quad (4\text{A})$$

where $p = p_x$ or p_y, as the case may be.

Seeing that the principal stresses act on planes perpendicular to each other, it is convenient to take the directions of the normals of these two planes as the axes of x and y.

To find the amount and direction of the stress on a plane whose normal is inclined to the axis of x at an angle θ in terms of the principal stresses.

This problem has already been solved on page 113, but it will now be solved by the same method as just used to find the principal stresses. Consider the equilibrium of a triangular wedge O A B (Fig. 64) as before, but in this case O B and O A are the planes of principal stress, their normals lying along the axes of x and y respectively, and A B is the plane whose normal

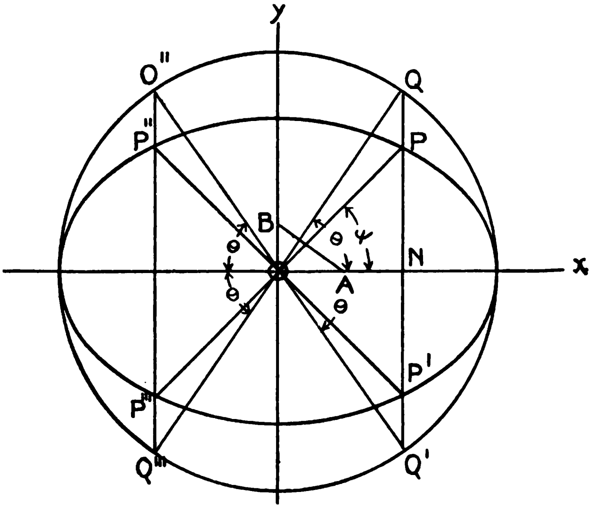


FIG. 64.

is inclined at an angle θ to the axis of x. Let ψ be the angle which the direction of the resultant stress on this plane makes with the axis of x, and let x, y be the co-ordinates of P, where O P represents the intensity of stress on the plane A B in magnitude and direction. It will be noticed that since the triangle O A B is infinitely small, A and B in the limit coincide with O.

Then for equilibrium

$$\text{OB} \times p_x = \text{AB} \times p \cos \psi, \text{ where } p = \text{OP},$$

and $$\text{OA} \times p_y = \text{AB} \times p \sin \psi.$$

Therefore $$p_x \cos \theta = p \cos \psi = x \quad \ldots \quad (5)$$

and $$p_y \sin \theta = p \sin \psi = y \quad \ldots \quad (6)$$

Squaring (5) and (6) and adding we get

$$\frac{x^2}{p_x^2} + \frac{y^2}{\quad} = \cos^2\theta + \sin^2\theta = 1,$$

which is the equation of an ellipse with major axis p_x and minor axis p_y, since p_x is taken as the greater principal stress.

Now $x = p_x \cos\theta$ and $y = p_y \sin\theta = p_x \sin\theta \times \frac{p_y}{p_x}$, or in words, the ordinate of the ellipse equals the ordinate at the same point (distant x from O) of the circle with centre O and radius p_x, multiplied by $\frac{p_y}{p_x}$. If, therefore, the ellipse with major axis $= p_x$ and minor axis $= p_y$ be drawn, and also the circle centre O and radius p_x, if any radius O Q of the circle be drawn and from the point Q a perpendicular be let fall on to the axis of x cutting the ellipse in P, O P will represent in magnitude and direction the intensity of stress on the plane whose normal is O Q. If Q N be produced to Q^1, cutting the ellipse again in P^1, O P^1 is the intensity of stress on the plane whose normal is O Q^1, making an angle $-\theta$ with the axis of x; it is equal to O P in magnitude, but its direction makes an angle $-\psi$ with the axis of x instead of $+\psi$.

It is obvious that if we take a radius of the circle inclined at an angle $\pi - \theta$, the magnitude of the intensity of stress on the plane whose normal it is will be again equal to O P in magnitude, and its direction will be the same as in the last case; and further, if the radius be taken making an angle $\pi + \theta$ the intensity of stress on the plane, whose normal that radius is, is still equal to O P, but its direction will coincide with that for the first case considered.

The intensity of stress on the plane A B may be obtained indirectly by drawing the triangle of forces acting on the wedge O A B (Fig. 65). In that figure O A^1 = O B $\times p_x$, O B^1 = O A $\times p_y$, $\therefore$ O P^1 the diagonal = A B $\times p$, $\therefore \frac{\text{O P}^1}{\text{A B}} = p$ the intensity of the stress on the plane A B, and the direction of O P^1 gives its direction. From equations (5) and (6) we get by squaring

$$p^2 (\cos^2\psi + \sin^2\psi) = p_x^2 \cos^2\theta + p_y^2 \sin^2\theta,$$

or $$p = \sqrt{p_x^2 \cos^2\theta + p_y^2 \sin^2\theta} \quad . \quad . \quad . \quad (7)$$

and by dividing equation (6) by equation (5) we get

$$\tan\psi = \frac{p_y}{p_x} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It will be noticed from (7) that when

$$\theta = 45 \text{ deg.}, \; p = \sqrt{\frac{p_x^2 + p_y^2}{2}};$$

also that p has the same magnitude for planes whose normals are inclined at angles θ, $\pi - \theta$, $\pi + \theta$, and $-\theta$, as already pointed out; and, moreover, its magnitude is independent of the signs of p_x and p_y. Again, from (8) it is clear that ψ has the same value for θ and $\pi + \theta$, which is equal and opposite to that for $\pi - \theta$ and $-\theta$, and it depends upon whether p_x and p_y are of the same or opposite signs.

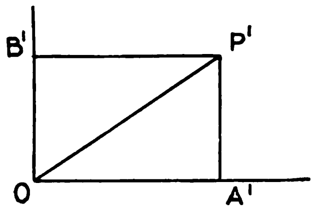


FIG. 65.

The intensity of normal stress on a plane whose normal is inclined at an angle θ to the axis of x.

The normal component of the stress intensity on the plane

$$= p\cos(\theta - \psi) = p(\cos\theta\ \cos\psi + \sin\theta\ \sin\psi).$$

From (5) and (6)—

$$= p_x \cos^2\theta + p_y \sin^2\theta.$$

If $\theta = 45$ deg. the normal component of stress $= \dfrac{p_x + p_y}{2}$.

The intensity of shearing stress on a plane whose normal is inclined at an angle θ to the axis of x.

The intensity of shearing stress on the plane $= p\sin(\theta - \psi)$

$$= p(\sin\theta\ \cos\psi - \cos\theta\ \sin\psi).$$

From (5) and (6)—

$$= p_x \sin\theta\ \cos\theta - p_y \sin\theta\ \cos\theta.$$

$$= \frac{p_x - p_y}{2} \times \sin 2\,\theta.$$

It will be noted from this that this shearing stress has the same numerical value for a plane inclined at an angle θ and an angle $\frac{\pi}{2} + \theta$.

This is a maximum when $\sin 2 = 1$, *i.e.* when $2\,\theta = 90$ deg. or 270 deg., or $\theta = 45$ deg. or 135 deg. Then the intensity of

$$\text{shear} = \frac{p_x - p_y}{2} = \frac{\sqrt{(p_n - p_n{}^1)^2 + 4p_t{}^2}}{2} \quad . \quad . \quad . \quad . \quad (9)$$

It therefore follows from (4) that the intensities of the principal stresses are equal to one-half the sum of the intensities of the normal stresses on two planes at right angles to each other $\pm$ the maximum intensity of shearing stress at the point. Since the sum of the normal intensities of stress on two planes at

right angles equals the sum of the principal stresses at the point this simply amounts to stating that $p_x, p_y = \frac{p_x + p_y}{2} \pm \frac{p_x - p_y}{2}$

Since p_x is the greater of the principal stresses (9) would have its greatest possible value if $p_y = -p_x$.

The plane of maximum shear does not coincide with the plane on which the direction of the resultant stress makes the maximum angle with the normal to the plane, as might at first sight be expected, and as the latter plane is of importance in connection with earth pressures it will be next considered.

To find the plane the stress on which makes the greatest angle with the normal to the plane, and the amount and direction of the resultant stress on such plane. This may be found very simply by considering two special cases of the ellipse of stress:—(*a*) If the two principal stresses are equal in magnitude and are of the same sign, it is obvious from (7) that the resultant stress on any other plane is of the same magnitude; and, moreover, equation (8) shows that the resultant stress in this case acts along the normal. (*b*) If the two principal stresses are equal in magnitude but opposite in sign, it still follows from (7) that the resultant stress on any other plane is of the same magnitude; but from (8) it will be seen that in this case the line of action of the resultant stress makes the same angle with the axis of x as the normal to the plane, but it lies on the opposite side of this axis, because $\tan \psi = -\tan \theta$ in this case.

It will be noticed that in these two cases the ellipse of stress coincides with the circumscribing circle.

It follows in the latter case that when $\theta = 45$ deg. the resultant stress lies in the plane—*i.e.* it is a shearing stress, therefore if the principal stresses are equal in magnitude but opposite in sign, the stresses on planes making an angle of 45 deg. with the planes of principal stress are pure shears; and *vice versa*, when the resultants on two planes at right angles are pure shears the stresses on planes making angles of 45 deg. with these planes are principal stresses equal in magnitude and opposite in sign.

Now, instead of finding the stress on a plane directly by the method of the ellipse of stress, these two propositions may be made use of, and as indicated above

$$p_x \text{ is identically equal to } \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2},$$

and $$p_y \text{ is identically equal to } \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2}.$$

The resultant of the first terms in the two identities may be first found, next the resultant of the second terms in the two identities ascertained, then the resultant of these two resultant stresses would be the actual resultant intensity of stress on the plane. It must be remembered that the ellipse of stress gives the *intensity* of the stress on a plane; if the intensities be multiplied by the area on which they act, taking the latter as of unit length perpendicular to the paper, it is only an application of the parallelogram of forces to find the resultant stress on a plane, and it may be done in the steps enumerated above.

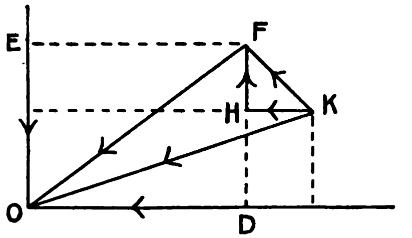


FIG. 66.

Thus, in Fig. 66 let O D represent $\frac{p_x + p_y}{2} \times$ O B, and let O E represent $\frac{p_x + p_y}{2} \times$ O A, then O F will represent the resultant of these two forces. Similarly, let H K represent $\frac{p_x - p_y}{2} \times$ O B, and F H represent $-\frac{p_x - p_y}{2} \times$ O A, then F K represents the resultant of these two forces. Now, the resultant of O F and F K = O K, and this is the actual resultant stress on A B, and $\frac{\text{O K}}{\text{A B}}$ would be the actual resultant intensity of stress on A B.

To obtain this intensity of stress directly by using the two above propositions (*a*) and (*b*). First draw O P′ (Fig. 67), the normal to the plane in question, and set off O P′ $= \frac{p_x + p_y}{2}$, then O P′ represents the intensity of stress on A B due to the two first terms of the identities, because they are two principal stresses equal in magnitude and of the same sign. From O draw

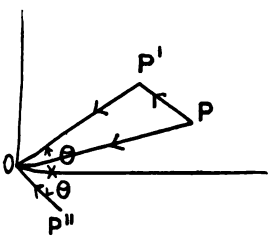


FIG. 67.

O P″, making the same angle as O P′ with the axis of x, but on the opposite side and equal in magnitude to $\frac{p_x - p_y}{2}$, then O P″ is the intensity of stress on A B due to the two second terms of the identities because they are two principal stresses equal in magnitude but of opposite sign. To combine O P′ and O P″, from P′ draw P′ P equal and parallel to O P″, then O P is the resultant intensity of stress on the given plane A B, and is, of course, of the same magnitude and direction as would have been obtained by drawing the ellipse of stress for the given principal stresses p_x and p_y; but this method of arriving at it enables an expression for the maximum value of the angle between O P, the resultant stress intensity, and O P′, the normal to the plane, to be readily obtained. For, the magnitudes of O P′ and P′ P are independent of the angle θ because they are respectively the semi-sum and the semi-difference of the principal stresses, therefore the maximum value of the angle P O P′ occurs when O P P′ is a right angle, because if O P′ be drawn (Fig. 68) equal to $\frac{p_x + p_y}{2}$ and a circle with centre P′ and radius P′ P $=$ $\frac{p_x - p_y}{2}$ be described, it is obvious that, if P be taken at any other point than where the tangent from O meets the circle, the angle P O P′ would be less. Therefore, from O draw the tangent O P to the circle and join P′ P; then O P P′ is a right angle, and to find the position of the axis of x draw O P″ parallel to P P′, *i.e.* making an angle of 90 deg. with O P, and bisect the angle P′ O P″. The bisecting line is obviously the axis of x. Thus the position of the normal O P′ to the plane on which the resultant stress O P makes a maximum angle P O P′ with the normal to the plane has been found, and the expression for its inclination is found as follows :—

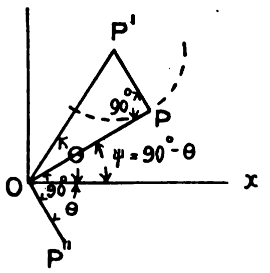


FIG. 68.

From (8)— $$\frac{\tan \theta}{\tan \psi} = \frac{p_x}{p_y}.$$

In this case from Fig. 68—

$$\psi = 90 \text{ deg.} - \theta \qquad \therefore \tan\psi = \cot\theta.$$

$$\therefore \tan^2\theta = \frac{p_x}{p_y}, \text{ or } \tan\theta = \sqrt{\frac{p_x}{p_y}}.$$

Again— $\sin \text{POP}' = \sin(\theta - \psi) = \frac{\text{PP}'}{\text{OP}'} = \frac{p_x - p_y}{p_x + p_y}.$

And the intensity of stress

$$= \text{OP} = \sqrt{\text{OP}'^2 - \text{PP}'^2} = \sqrt{\left(\frac{p_x + p_y}{2}\right)^2 - \left(\frac{p_x - p_y}{2}\right)^2} = \sqrt{p_x p_y}.$$

Thus the inclination of the normal of the plane when the angle $\theta - \psi$ is a maximum is $\tan^{-1}\sqrt{\frac{p_x}{p_y}}$, and the resultant stress on the plane $= \sqrt{p_x p_y}$.

CHAPTER VII

BEAMS AND GIRDERS WITH SOLID WEBS

In this chapter the stresses acting on the cross-section of a beam or girder with solid web, which resist the bending moment and shearing force, will be considered. The stresses normal to the vertical longitudinal planes of the girder may be neglected, this plane being a plane of symmetry with respect to the stresses, so that it is only necessary to consider the stresses on planes normal to this plane. It must be borne in mind that in engineering structures the stress is always kept within the elastic limit. Since at any point in a beam where there is a stress a corresponding strain is thereby induced, and since the beam bends without altering the length in the neutral plane, and so that the strains or elongation and shortening of the fibres are proportional to their distance from the neutral plane, it follows that the intensities of stress normal to vertical cross-sections are also proportional to the distance of the point considered from the neutral plane. Thus if a be the intensity of stress normal to the cross-section at unit distance from the neutral plane, $a\,y$ is the intensity of stress at distance y from this plane. Therefore the intensity of the stress acting on a thin strip of the cross-section parallel to the neutral plane will be of constant intensity, and the moment of the stress on this strip of area will be proportional to the square of the distance from the neutral plane and equals $a\,y^2\,\delta\,\text{A}$, where $\delta\,\text{A}$ is the area of the strip parallel to the neutral plane. Therefore the moment of the stress over the whole area $= a\,\overline{y^2}\,\text{A}$, where $\overline{y^2}$ is the mean square of the distance from the neutral plane or the radius of gyration squared, which is denoted by k^2, and A is the area of the cross-section.

$$\therefore\ a\,\overline{y^2}\,\text{A} = a\,k^2\,\text{A} = \frac{f}{y} \times \text{I},$$

where f is the normal stress at distance y from the neutral plane and $\text{I} = k^2\,\text{A}$ is the moment of inertia of the cross-section about the neutral plane.

Since the bending moment M balances the moment of stresses at right angles to the section, we have

$$\text{M} = \frac{f}{y}\text{I} \quad \text{or} \quad \text{M} = f\,\text{Z}, \ \therefore \ f = \frac{\text{M}}{\text{Z}}$$

where $\text{Z} = \frac{\text{I}}{y}$, and therefore depends entirely on the shape of the cross-section and is called the modulus of the section.

It is sometimes convenient to take the moment of the tensile stresses separate from the moment of the compressive stresses; if A_c is the area above the neutral plane in compression and k_c is the radius of gyration for that area, and A_t and k_t are the corresponding quantities for the area below the neutral axis in tension, then—

The moment of the stresses normal to the section

$$= a\,k_c^2\,\text{A}_c + a\,k_t^2\,\text{A}_t,$$

$$\therefore \ \text{M} = a\,\text{I}_c + a\,\text{I}_t \quad . \quad . \quad . \quad . \quad (1)$$

where I_c and I_t are the moments of inertia of the portions of the cross-section in compression and tension respectively.

Intensity of Shearing Stress.

To find the intensity of shearing stress at any point in the vertical section distant y from the neutral plane, we have to take into account that the intensity of shearing stress in the horizontal plane at the point has the same value, and the latter can be easily deduced.

In Fig. 69 let the axis of x be in the neutral plane, and let E be the point at which the intensity of shearing stress is required,

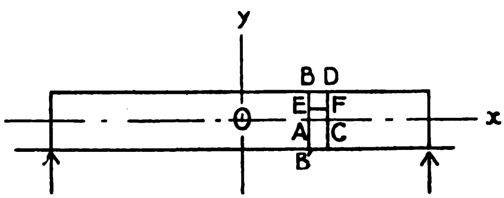


FIG. 69.

and let its co-ordinates be x, y, let A B the distance from the neutral plane of the extreme fibre equal y_1 and $\text{AB}^1 = y_2$. The section C D is taken near to the section A B at a distance δx from it. Let f be the intensity of stress at E normal to A B, and q be the intensitv of shearing stress there, and let p be

[illegible] from the neutral [illegible]

$$\frac{f}{\eta} = \frac{M}{I}, \text{ and } f = \frac{M}{I}\,\eta.$$

[illegible] between E and B

$$= \int [illegible] = \frac{M}{I}\int [illegible]\, z\,\eta\, d\,\eta \quad . \quad . \quad . \quad . \quad (2)$$

where [illegible] above the neutral plane.

[illegible] E F D is pressed [illegible] the section [illegible] A B over its [illegible] where F is the [illegible] the horizontal force [illegible] must balance

$$= \frac{[illegible]}{I}\int [illegible]\, z\,\eta\, d\,\eta = \frac{F\,[illegible]}{I}\int [illegible]\, z\,\eta\, d\,\eta \quad . \quad . \quad . \quad (3)$$

[illegible] shearing stress on E F we must [illegible] where b is the breadth [illegible] Therefore—

$$q = \frac{F}{I\,b}\int_{y}^{y_1} z\,\eta\, d\,\eta.$$

The [illegible] shearing stress, q_1, say, is its [illegible] where the breadth $= b_1$, say.

Therefore—

$$q_1 = \frac{F}{I\,b_1}\int_{0}^{y_1} z\,\eta\, d\,\eta.$$

The area of the cross-section at A B $= S = \int_{y_2}^{y_1} z\,d\,\eta$, and the mean intensity of shearing stress at the section $= \dfrac{F}{S}$. Therefore the ratio of the maximum to the mean

$$= \frac{q_1\,S}{F} = \frac{S}{I\,b_1}\int_{0}^{y_1} z\,\eta\,d\,\eta,$$

which for a rectangle equals $\frac{3}{2}$, and for an ellipse or circle equals $\frac{4}{3}$.

Next consider the case of a deep girder, such as a plate girder. Let Fig. 70 represent a portion of the elevation of such a girder and Fig. 71 its cross-section; Fig. 72 is a diagram of intensity of normal stress on the section, f_c being the stress intensity at

the top of the upper flange and f_1 its intensity at the bottom of this flange. In Fig. 73 the area down to any horizontal line represents the total normal stress on the area down to the line, because $\text{A F} = f_c \times \text{B}$, $\text{E G} = f_1 \times \text{B}$, $\text{E H} = f_1 \times t$, corresponding to the cross-section X Y in Fig. 70; A f, E g, and E h represent the corresponding quantities for the section $\text{X}^1\ \text{Y}^1$

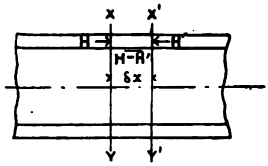


FIG. 70.

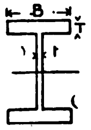


FIG. 71.

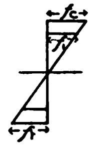


FIG. 72.

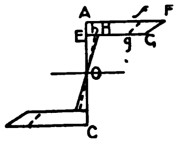


FIG. 73.

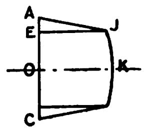


FIG. 74.

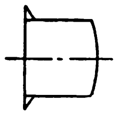


FIG. 75.

in Fig. 70, distant δx from X Y. Using the notation employed for a beam above, the area of the diagram in Fig. 73 down to a horizontal line distant y from O $= \int_y^{y_1} p\, z\, d\,\eta$, which is the total stress down to the line distant y from O.

The area F G $g\, f$ is the difference between the total normal stress H in the flange at section X Y, and the total normal stress H^1 in the flange at section $\text{X}^1\,\text{Y}^1$. Let this amount $\text{H} - \text{H}^1$ be plotted at E J (Fig. 74), and make O K in the same figure equal E J plus the area of the triangle H O h (Fig. 73), the intermediate abscissæ being the area of F G $g\, f$ and H O h down to the level considered. Thus the abscissæ in Fig. 74 represent the total shear between the sections X Y and $\text{X}^1\,\text{Y}^1$ at their particular levels. In Fig. 75 these abscissæ are divided by the breadth B of the flange and t of the web, so that the abscissæ of Fig. 75 represent the intensities of shearing stress at the different depths; and it will be observed that the intensity of shearing stress in the flanges is very small. It is obvious, therefore, that in the case of a plate girder most of the normal stress will be taken up in the flanges represented by the area A F G E (Fig. 73), that taken up by the web E H O being small in proportion as the area

of the web is small and the intensity of stress in the web gradually decreases to zero at the neutral plane. Thus, if the help of the web be ignored, the effect will be to make the flange rather too large, and the deviation from exactness is on the side of safety.

In that case, $I_c = A_c Y_c^2$, and $I_t = A_t Y_t^2$, Y_c and Y_t are the distances from the neutral plane to the centres of the flanges, the moment of inertia about the horizontal lines through the centres of gravity of the flanges, which is small, being neglected.

Now from (1) $M = I_c \frac{F_c}{Y_c} + I_t \frac{F_t}{Y_t}$ where F_c and F_t are the intensities of stress at the centres of the flanges. Substituting the above value for I_c and I_t, $M = A_c Y_c F_c + A_t Y_t F_t$.

But $A_c F_c = A_t F_t = H$, the total stress in the compression or tension flange at the section.

$$\therefore \quad M = H(Y_c + Y_t) = H \times D \quad . \quad . \quad (4)$$

where $D = Y_c + Y_t$ the depth from centre to centre of the flanges.

It will be observed that in making use of this formula we take the working intensity of stress equal to the average stress in the flange, instead of equal to the stress at the outside fibre, in order to counteract the effect of ignoring the help of the web and neglecting the moment of inertia of the flange about the horizontal line through its centre of gravity.

Shearing Stress.—From Fig. 74 it is obvious that if the web is deep, compared with the flanges, that by far the greater part of the shearing stress is taken by the web, and if it be assumed to take it all, the defect from exactness would be on the safe side. Referring to Fig. 70, the intensity of shearing stress in the upper flange at its lower surface $= \frac{H - H^1}{B\,\delta x}$, whereas in the web at the same level the intensity of shearing stress $= \frac{H - H^1}{t\,\delta x}$ —*i.e.* it is greater in the web in the proportion of $B : t$. Fig. 75 shows that the intensity of shear in the web increases slightly towards the neutral axis, but neglecting this slight increase and taking the intensity of shear in the web to be constant, as δx becomes infinitely small the intensity of shearing stress in the web

$$= \frac{d H}{d x} \times \frac{1}{t},$$

substituting from (4) $= \frac{d M}{d x} \times \frac{1}{D \times t} = \frac{F}{D \times t} \quad . \quad . \quad (5)$

where F is the shearing force at the section as before. This -iously also the shear in the web in the vertical plane.

In applying the formula $M = H \times D = A \times f \times D$, where f is the working intensity of stress, in order to avoid useless refinement, D, if great compared with the thickness of the flange at the centre, may be taken as constant and equal to the distance from outside to outside of the first plate of the flanges next the connecting angles, and this plate is generally carried to the ends of the girder. The horizontal flanges of the connecting angles should be taken as part of the flanges. If the flange is thick a more accurate value of the depth must be taken, unless the girder is proportionally deep. For the tension flange the area removed in the cross-section for the rivets must be added to the net area as calculated from the above formula to arrive at the necessary size of plates; but for the compression flange this is not necessary, as the rivets are supposed to fill the holes and to be capable of taking compression.

In using (5) to calculate a value for the thickness of the web, D should really be the distance from centre to centre of the rows of rivets in the vertical flanges of the connecting angles, and the working intensity of shear must be kept down below that allowed for the rivets, or, as will be shown, the tension and compression in the web attain considerably higher intensities than in the flanges. The thickness of the web has also to be regulated in order to keep down the bearing pressure on the connecting rivets. It is generally not desirable to make the thickness less than $\frac{3}{8}$ inch, even if calculation should show that this is permissible, otherwise the effect of corrosion, should it take place, would cause a large reduction of available area.

From the considerations in Chapter VI. it will be evident that the vertical and horizontal planes are not the planes of principal stress, therefore the normal stress on a vertical section is not the maximum intensity of stress at a point. But in the flange, as has already been shown, the intensity of shearing stress is very small, therefore the normal stress on the vertical plane in the flanges differs very little from the maximum, and one plane of principal stress very nearly coincides with it. In the web, however, particularly towards the ends of the girder, the intensity of shearing stress is very appreciable. At the neutral axis the direct stress is zero, and there is only shear on the vertical and horizontal planes there, therefore the planes of principal stress there would be at angles of 45 deg. with the vertical and horizon[illegible]es. and the intensity of tension or compression would [illegible] that of the shearing stress. At

the upper and lower extremities of the web the intensity of normal stress is evidently the same as in the flanges at these levels, consequently in the compression portion of the web the greater principal stress will be a compression and greater than the intensity of compression in the flange, and will act on a plane inclined to the vertical. A similar consideration applies to the tension portion of the web. This is evident from equations (4A), Chapter VI., which show that in this case the maximum principal stress $= p = \frac{p^1{}_n \pm \sqrt{p^1{}_n{}^2 + 4p^2{}_t}}{2}$, the second term being taken as the same sign as $p^1{}_n$; and the cotangent of the inclination of its normal to the horizontal $= -\frac{p}{p_t}$.

For example, if the compressive stress in the flange is 5 tons per square inch, and the intensity of shear in the web at that section in the vertical or horizontal plane is 3 tons per square inch, then

$$p = \frac{5 + \sqrt{25 + 36}}{2} = \frac{5 + \sqrt{61}}{2} = \frac{12{\cdot}8}{2} = 6{\cdot}4 \text{ tons per sq. inch,}$$

and $$\cot \theta = -\frac{6{\cdot}4}{3} = -2{\cdot}1.$$

The other principal stress $= \frac{5 - 7{\cdot}8}{2} = -1{\cdot}4$ tons per sq. inch.

The maximum intensity of shearing stress is on the plane bisecting the angle between the two principal planes, and equals $\frac{p_x - p_y}{2} = \frac{6{\cdot}4 + 1{\cdot}4}{2} = 3{\cdot}9$ tons per square inch, which shows that the maximum intensity of shearing stress is appreciably greater than its intensity on the horizontal and vertical planes.

At the neutral plane, where the intensities of the principal stresses is equal to that of the shearing stress, their value would be 3 tons per square inch. The planes of principal stress can thus be found at any point in the web, as indicated in Fig. 76, for a uniform load. It is obvious, therefore, that if the intensity of direct stress in the flange is made equal to the working intensity of stress towards the ends of the girder, the maximum intensities of compression and tension in the web are bound to exceed those in the flange, and unless the intensity of shearing stress in the web is made small, they would largely exceed the flange stress. The only way to avoid this is to reduce the working intensity of flange stress towards the ends of the girder as the intensity of

shear in the web increases—*i.e.* to decrease p^1_n in the formula above so as to keep p constant, as p_t increases—instead of proportioning the flange area by the formula $M = f \times A \times D$; or else to restrict the increase by keeping the intensity of shearing stress low. It will also be seen shortly that it is necessary to keep this intensity of shear low to prevent the intensity of bearing pressure on the rivets connecting the web to the flange angles exceeding suitable limits. When the girder is deep, in order to prevent

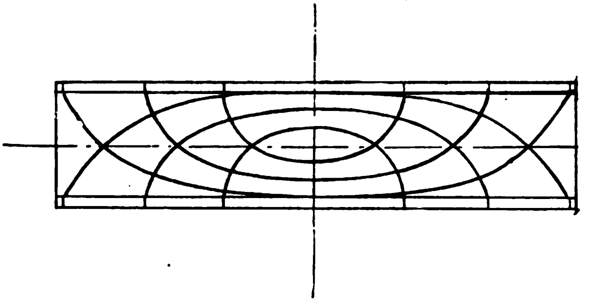


FIG. 76.

the comparatively thin web from buckling, owing to the compressive stresses present in it, it appears desirable from the analogy of the long column to stiffen it. The rules generally given for effecting this purpose are unsatisfactory, but it is easy to see that, if the web is of constant thickness and the stiffeners are of constant cross-section, they should be placed nearer together, near the ends of the girder. The object of the stiffeners is simply to prevent the web buckling, the stresses being resisted as explained above; if the stiffeners and part of the web on either side of them are sufficient to take the shearing force at that section as a compression, the portion of the web between the foot of that stiffener and the top of the next nearer the end would transmit the shear as a tension to the next stiffener, so that if a tendency to collapse developed the method of resisting the bending moment and shearing stress would be altered, the girder becoming more of the type of an N girder already considered. If it cannot collapse when considered in this manner it is perfectly safe; that is to say, if the stiffener with the help of the web on either side of it is strong enough when treated as a strut to take the shearing force at the section, and the web is thick enough to transmit this as a tension to the next

stiffener, there is no fear of the girder buckling. Now the total tension in the web between the foot of one stiffener and the top of the next nearer the end equals the shearing force at the first × cosec θ, where θ is the inclination to the horizontal of the diagonal of the bay between the two stiffeners considered. As the first term in this expression increases to a maximum as the end is approached, the second term—*i.e.* cosec θ—should diminish if the thickness of the web and stiffeners are uniform, and to effect this the distance apart of the stiffeners must decrease as

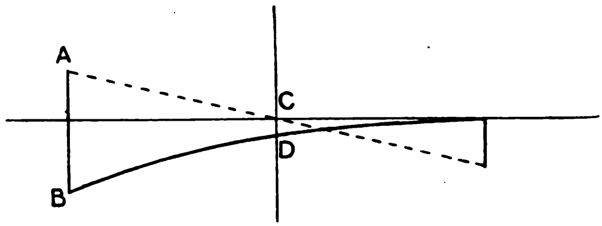


FIG. 77.

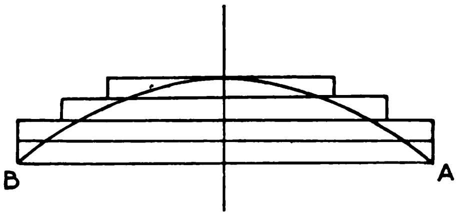


FIG. 78.

the ends of the girder are approached. Fig. 77 shows the combined shear for a uniform dead and moving load; the stiffeners in this case should be nearer together at the ends than at the centre in the ratio of C D : A B. This is reduced in practice by the fact that the thickness of the web at the centre is made greater than is necessary for strength.

In a plate girder to find the number of rivets necessary to attach the web to the flange angles.

When the shearing force F at any vertical section is known, since the intensity of the shearing stress over the web is taken as constant, $\frac{F}{D \times t}$ is the intensity of shear in the web in the vertical and horizontal directions. In 1 foot length of the web the shear

would be $\frac{F}{D \times t} \times 12\,t = \frac{12\,F}{D}$ if D is in inches, or $= \frac{F}{D}$ if D is in feet.

Now the rivets connecting the web to the flange angles are in double shear, therefore the number of rivets n in one foot length must be strong enough to resist a shear of this amount —*i.e.* $n \times \frac{\pi\,d^2}{2} \times f_s$, must be not less than $\frac{F}{D}$, where D is in feet, f_s being the working intensity of shearing stress allowed in the rivets and d their diameter; also the rivets must be numerous enough to prevent the bearing pressure between the rivets and the web being excessive—*i.e.* if N be the number of rivets per foot required for this purpose, $N \times d \times t \times f_b$ must not be less than $\frac{F}{D}$, where f_b is allowable intensity of bearing pressure. For large rivets in double shear the fulfilment of the latter condition requires a greater number of rivets than the former, as will be seen by working out the values of n and N from the above expressions in any particular case.

$$\therefore\ N \times d \times t \times f_b = \frac{12\,F}{D}, \text{ or } N = \frac{12\,F}{D \times d \times t \times f_b},$$

the dimensions being in inches.

If this gives a number of rivets greater than can be conveniently got in, the thickness of the web must be increased in order to keep the number within convenient limits.

The flanges of the girder would generally consist of a number of separate plates, and since the area necessary as the bending moment decreases is less, the plates may be stopped before reaching the ends of the girder.

For example, if the curve (Fig. 78) represents the bending moment diagram, and it is found that at the centre three plates are required in addition to the horizontal flanges of the angles to give the necessary area for, say, the compression flange, the centre vertical should be divided into four parts, the lowest representing the moment of resistance of the angles and the three upper parts representing the moment of resistance of the three plates. It is generally sufficient to divide it in proportion to the cross-sectional areas of the angles and plates. The plates should be taken beyond the curve a distance to allow a sufficient number of rivets to be inserted to effectually connect the plates before reaching the section. It is not usual to carry them further

than this for the purpose of limiting the maximum tension and compression in the web to the value of the working intensity of stress in the flanges.

In designing the cross-girders for a bridge the maximum wheel loads that can come on to a bay length must be taken and considered to be placed in the position that will cause the maximum possible portion of such load to come on one cross-girder. For a double line of rails, a similar load must be taken on the other track at the same time. Thus the cross-girders are subjected to four concentrated loads at the position of the rails, in addition to the dead load, which may also be taken as acting at the same points without introducing much error, as the portion of the dead load communicated to it by the stringers is, of course, brought on in those positions. The depth of the cross-girders for a double track should be not less than 2 feet, unless circumstances render such a depth impossible.

In the case of the stringers or rail-bearers, to find the maximum flange stress, the maximum load that the axles of the locomotives can impose on a bay length must be considered as acting in the position that will produce the maximum bending moment at the centre of the stringer; and to find the maximum shear the loads must be placed in the position to give the greatest shear at the ends of the stringers. It is best not to consider the stringers as girders fixed at the ends, although their connection to the cross-girders no doubt partially introduces this condition.

Cross-girder.

As an example, take the case of a cross-girder and rail-bearer for the bridge for which the stresses in the members have been found using the same typical load.

It is obvious from Fig. 79 that practically the maximum load that can come on to a cross-girder is indicated at the cross-girder B in that figure, with the disposition of load there shown, where the load transmitted to B can be found by taking moments about A and C of the loads on the spans A B and B C respectively. The load on the cross-girder B

$$= 2 \times 15 \times \frac{17 \cdot 5}{20} + 2 \times 15 \times \frac{12 \cdot 5}{20} + \frac{1 \cdot 2 \times (7 \cdot 5)^2}{2 \times 20} + \frac{15 \times 2 \cdot 5}{20}$$

$= 45 + 1 \cdot 7 + 1 \cdot 9 = 48 \cdot 6$ tons.

This is the maximum live load brought by each track on to the cross-girder, and using the third method in Chapter V.

for determining the cross-sectional area of members, this may be at once doubled—*i.e.* it may be taken as the load communicated to the cross-girder at each rail. Now the platform load equals $20 \times 0{\cdot}7 = 14$ tons, and if this is taken as being applied at each rail it gives $\frac{14}{4} = 3{\cdot}5$ tons at each rail. The total equivalent load at each rail as indicated in Fig. 80 is therefore 52 tons. The bending moment will be a maximum between the two

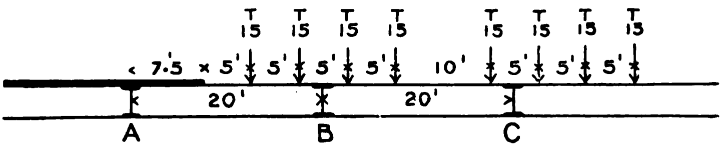


FIG. 79.

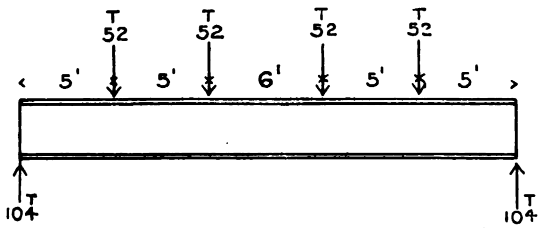


FIG. 80.

middle rails, where it equals $104 \times 10 - 52 \times 5 = 780$ tons-feet. Take the depth of the cross-girder as 3 feet, and let angles $4\frac{1}{2} \times 4\frac{1}{2} \times \frac{1}{2}$ be used to connect the web to the flange.

For the area A_c of the compression flange, since it is not necessary to allow for material removed for rivet holes, we have

$$A_c \times 7 \times 3 = 780,$$

$$\therefore A_c = 37{\cdot}1 \text{ square inches.}$$

Now A_c = area of horizontal flanges of angles + width of plates $\times$ T, where T is the thickness of the plates in the flange; let the width of the flange be 18 inches. The area of the horizontal flanges of the angles $= (4\frac{1}{2} + 4\frac{1}{2}) \times \frac{1}{2} = 4\frac{1}{2}$ square inches.

$$\therefore A_c = 37{\cdot}1 = 4{\cdot}5 + 18 \times T.$$

$$\therefore T = 1{\cdot}8 \text{ inches.}$$

To find the area A_T of the tension flange, it must be remembered that the material removed for rivet holes must be allowed for, and by finding the thickness of the compression flange

first a sufficiently near approximation to the length of the rive is readily obtained, say $2\frac{1}{2}$ inches. If rivets $\frac{7}{8}$-inch diamet are used, the flange area would be reduced by the longitudin section of two rivets—*i.e.* $2 \times 2\frac{1}{2} \times \frac{7}{8} = 4{\cdot}5$ square inches, say Now the area required is given by

$$A_T \times 9 \times 3 = 780, \quad \text{or} \quad A_T = 29 \text{ square inches.}$$

Or adding the area removed for rivets $= 33{\cdot}5$ square inches. The horizontal flanges of the angles contribute $4\frac{1}{2}$ square inches of this, and leave 29 square inches as the area of the plates. The thickness of the plates therefore is $\frac{29}{18} = 1{\cdot}6$ inches.

Of course, the nearest market sizes that give the necessary thickness would be chosen.

Thickness of web. The intensity of shear in the web should be kept low in order to avoid high tensile and compressive stresses in the web near the ends, and for this purpose one or more of the flange plates should be carried on to the ends of the girder in addition to the connecting angles.

Suppose the maximum intensity of shear in the vertical or horizontal plane in the web is limited to 4 tons per square inch. The depth between centre line of rivets connecting the web to the upper and lower flanges respectively is 30 inches, therefore the thickness of the web t is obtained as follows :—

$$30 \times t \times 4 = 104, \quad \text{or} \quad t = 0{\cdot}87 \text{ inch.}$$

Now the least thickness of plate that will develop the full working intensity of shearing strength of a $\frac{7}{8}$-inch rivet in double shear, taken as 6 tons the square inch, without the bearing intensity of pressure between the rivet and plate exceeding 12 tons per square inch, is found by equating the two—

$$\frac{7}{8} \times t \times 12 = 2 \times 0{\cdot}6 \times 6$$

where $0{\cdot}6$ is the area of a $\frac{7}{8}$-inch rivet;

$$\therefore t = \frac{4{\cdot}8}{7} = 0{\cdot}7 \text{ inch.}$$

That is to say, with these unit stresses, if the thickness of a plate is less than $0{\cdot}7$ inch, the number of connecting rivets, if in double shear, must be designed to limit the bearing pressure to the unit stress, and the allowable unit shearing stress will not be reached ; but if the thickness is greater than $0{\cdot}7$ inch, the number of rivets must be designed to limit the shearing stress to the unit stress chosen.

Rail-bearer or stringer. The position of the load that will give the maximum bending moment at the centre of the stringer

is when four axles are symmetrically placed with respect to the centre. If the full axle load be taken as acting on such stringer it will have the effect of doubling the live load. Thus we have on the stringer on each side of the centre 15 tons, 2 feet 6 inches from the centre, and 15 tons, 7 feet 6 inches from the centre. The platform load will be less than $\frac{0{\cdot}7 \times 20}{4}$, say 3 tons, because the weight of the cross-girder is excluded. The bending moment at the centre therefore is $30 \times 10 - 15 \times 7{\cdot}5 - 15 \times 2{\cdot}5 + 3 \times \frac{20}{8} = 157\frac{1}{2}$ tons-feet. The flange areas are found as before.

The maximum shear will occur at the end of the stringer when the leading wheel of the four is just approaching the end of the stringer; the reaction and therefore the shearing force at the end then is $15 + \frac{15}{20}\,15 + \frac{10}{20}\,15 + \frac{5}{20}\,15 = \frac{5}{2} \times 15 = 37\frac{1}{2}$ tons; the web thickness t taking the same unit shearing stress of 4 tons per square inch is obtained as before—*i.e.*

$$30 \times t \times 4 = 37\tfrac{1}{2}, \quad \text{or} \quad t = 0{\cdot}3 \text{ inch.}$$

As it is not desirable to make it less than $\frac{3}{8}$ inch, this thickness would be selected.

It will be observed that since the thickness is less than 0·7 inch the connecting rivets must be designed to limit the bearing pressure. T section stiffeners would be used at about four points in the length of the stringers, between stringer and stringer and between the outside stringers and the main girder flanges, to diminish vibration and prevent the stringers buckling.

CHAPTER VIII

CONTINUOUS GIRDERS AND CANTILEVER BRIDGES

WHEN a girder is continuous over one or more intermediate supports, the problem differs from the case of a girder supported at the ends in the fact that there are bending moments at the intermediate piers over which the girder is continuous, and this fact alters the relative values of the reactions on the piers. Any single span is therefore in equilibrium under the action of the loads it carries, the reactions due to that load at the piers and the stress couples equal to the bending moments over the piers. The portions of the two reactions on the piers at the extremities of a span, due to the load upon it, must be equal to the load on the span, because the vertical forces must balance, as a couple cannot balance a force. But the reaction at such intermediate pier is due partly to the load on the span on one side of it, and partly due to the load on the span on the other side of it, therefore to find the total reaction at a pier these partial reactions must be determined and added together.

The effect of a stress couple being present over a pier can be realised if a girder supported at the two ends, but prevented from moving endwise, is supposed to have a vertical lever rigidly secured to one end (Fig. 81), to the extremity of which a horizontal force is applied acting, say, in a direction away from the centre of the girder; this will introduce a couple at the section of the girder at the end B,

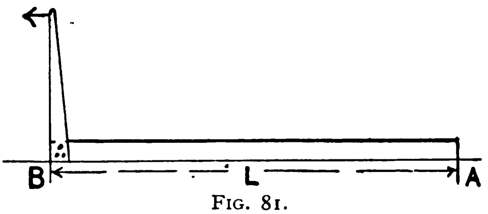


FIG. 81.

which will obviously tend to lift the end A of the girder, and therefore to reduce the reaction there compared with its value when the couple is not acting. The reaction at B will be increased by the same amount as that at A is diminished,

because the sum of the two must still equal the load on the girder. If, therefore, R is the reaction at A when no couple is applied at B, and R^1 is the reaction there when the couple is acting, the value of the couple will be $(R - R^1) \times L$. Thus, if a girder consists of two spans, A B, B C (Fig. 82), and it is continuous over the centre pier B, and a load is placed, say, near the centre of the span A B only, it will tend to lift the free end C of the unloaded span unless that is held down; neglecting the

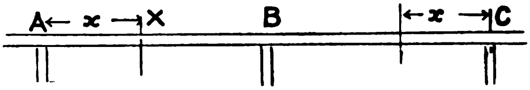


FIG. 82.

weight of the beam, the force required to hold it down $\times$ B C equals the stress couple over the pier B, because the reaction at C is zero if the beam is severed at B, and the reaction on this pier will be increased by the amount of the holding down force at C. At the piers over which the girder is continuous, the upper fibres will tend to be in tension, and therefore the lower

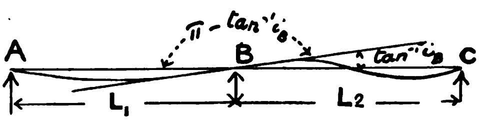


FIG. 83.

ones in compression, because the load on the span on either side of it will cause that part of the girder to bend, so as to be convex upwards. This shows that the bending moment at such a pier is negative. The tangent to the deflected curve of the centre line over such a pier would not generally be horizontal unless the spans and loads are symmetrically disposed relatively to the pier in question.

Theorem of Three Moments.

When the load is uniformly distributed, as is generally assumed to be the case for the dead load, the bending moments at the piers, and therefore the reactions there, can be found from the theorem of three moments, which gives an equation connecting the bending moment at each three consecutive piers. Since the couples acting at the ends of a span where the girder is continuous over the piers are exerted by the stresses in the flanges of the adjacent spans, it is clear that the values of these moments

cannot be obtained from statical considerations, but must be evolved from the elastic properties of the material of the girder.

It is easy to write down the bending moment at any section in terms of the load on the portion of the span between the section and the next pier, the stress couple over that pier and the reaction at it due to the load on the span in which the section is taken. And it is known from the elastic properties of a beam that the change of inclination, $i_2 - i_1$, of the axis of the beam between the two sections distant x_2 and x_1 from the piers $= \int_{x_1}^{x_2} \frac{\text{M}\, d\, x}{\text{E I}}$. If the load is uniformly distributed, the expression for M is continuous, but for a concentrated load it is discontinuous at the load, the expression for the bending moment on one side of the load being different from that on the other side. If, therefore, the concentrated load comes in the length in which the total change of inclination is required, the integration has to be performed for the length from x_1 to the load and for the length from the load to x_2 separately. It will be evident that if i_2 is the inclination of the centre line over one pier, it is necessary to eliminate i_1 before it can be utilised ; i_1 may be the inclination of the centre line at the preceding pier, and to find its value it is necessary from the expression for the inclination i to find that for the deflection u. Now from the elastic properties of the beam $u_2 - u_1 = \int_{x_1}^{x_2} i\, d\, x$, and, since the level of the centre line at the piers is unaltered, u is known for these points, and consequently an equation for i_1 is obtained.

Thus an expression is obtained for the inclination of the centre line at a pier by considering the deflection of the span on one side of it ; similarly, an expression for this inclination of the centre line at the same pier may be obtained from a consideration of the deflection of the span on the other side of the pier. By substituting the values of the pier reactions in terms of the bending moments at the piers, and equating these two expressions for the inclination of the centre line at the pier, a[n] equation involving the moments at three consecutive piers i[s] obtained.

Let A B and B C (Fig. 82) be two consecutive spans, and le[t] the girder be continuous over the three piers, A, B, and C.

It must be borne in mind that the symbol representing th[e]

bending moment at any section is the couple exerted by the part of the beam on the right of that section on the portion to the left of it, and therefore the action of the latter on the former, which is equal and opposite, would be represented by the same symbol but preceded by the minus sign.

We will suppose that the piers are level, and that the cross-section of the girder is constant, and that the load on each span is uniformly distributed, but not necessarily of the same intensity on different spans. We will call the reaction at the pier B due to the load on A B, R_{BA}, and the reaction at B due to the load on B C, R_{BC}, the first letter of the suffix denoting the pier considered and the two letters of the suffix denoting the span. The total reaction at B, $R_B = R_{BA} + R_{BC}$, and similarly for the other piers. Let L_1 be the length of the span A B and w_1 the uniform load per linear foot on it. Take A as the origin and the axis of x in the direction A B, and let x be the distance from A of the section considered. The bending moment at X = M $= M_A + R_{AB} \times x - \frac{w_1 x^2}{2}$, by taking moments about X. Now the alteration in the inclination of the centre line at X compared with its inclination at A

$$= i - i_A = \int_0^x \frac{M}{EI} dx$$

$$= \frac{1}{EI} \int_0^x \left(M_A + R_{AB} \times x - \frac{w_1 x^2}{2}\right) dx,$$

$$\therefore i - i_A = \frac{1}{EI}\left(M_A \times x + R_{AB} \times \frac{x^2}{2} - \frac{w_1 x^3}{6}\right).$$

In particular when

$$x = L_1,\ i = i_B,$$

$$\therefore i_B - i_A = \frac{1}{EI}\left(M_A \times L_1 + R_{AB} \times \frac{L_1^2}{2} - \frac{w_1 L_1^3}{6}\right) \quad . \quad . \quad (1)$$

To find i_A, we know that the deflection at the piers A and B is zero, therefore finding an expression for the difference of the deflection u between B and A from that for i we have

$$u_B - u_A = \int_0^{L_1} i\, dx = \int_0^{L_1} \left\{ i_A + \frac{1}{EI}\left(M_A \times x + R_{AB} \times \frac{x^2}{2} - \frac{w_1 x^3}{6}\right)\right\} dx.$$

Since $u_B = u_A = o$, we have

$$i_A L_1 + \frac{1}{EI}\left(M_A \frac{L_1^2}{2} + R_{AB}\frac{L_1^3}{6} - \frac{w_1 L_1^4}{24}\right) = o,$$

or
$$i_A = -\frac{1}{EI}\left(M_A \frac{L_1}{2} + R_{AB}\frac{L_1^2}{6} - \frac{w_1 L_1^3}{24}\right).$$

Substituting this value for i_A in (1), we have

$$EI\,i_B = \tfrac{1}{2} M_A \times L_1 + \tfrac{1}{3} R_{AB} L_1^2 - \frac{w_1 L_1^3}{8} \quad . \quad . \quad (2)$$

Now R_{AB} can be found in terms of the moments at A and B by taking moments about B, when we obtain

$$M_A + R_{AB} \times L_1 - \frac{w_1 L_1^2}{2} - M_B = 0, \text{ or } R_{AB} = \frac{M_B - M_A}{L_1} + \frac{w_1 L_1}{2}.$$

Substituting this value for R_{AB} in (2), we get—

$$EI\,i_B = \tfrac{1}{6} M_A L_1 + \tfrac{1}{3} M_B L_1 + \tfrac{1}{24} w_1 L_1^3 \quad . \quad . \quad . \quad (3)$$

Now instead of taking A for the origin take C for it, and the axis of x in the direction C B, and let x be the distance from C of the section considered. L_2 is the length of B C and w_2 the intensity of the uniform load upon it. With A as origin, we obtain the angle the tangent to the centre line at B (Fig. 83) makes with A C, the angle being measured by the angle turned through by the line A C, if we suppose it to rotate about B in the contra-clockwise sense, since the axis of x is drawn to the right from A, till it coincides with the tangent to the centre line at B. This angle is $\tan^{-1} i_B$. Now taking C as origin, we obtain the angle which the tangent to the centre line at B makes with C A—*i.e.* the angle through which C A must rotate about B in the clockwise sense (since the axis of x is drawn to the left from C) till it coincides with the tangent at B—this angle is obviously $\pi - \tan^{-1} i_B$. If, therefore, we go through the work exactly as before, we shall obtain an equation similar to (3), but since $\tan(\pi - \tan^{-1} i_B) = -i_B$, on the left we shall have $-EI\,i_B$; on the right we must change A into C, L_1 into L_2, and w_1 into w_2; we then get—

$$-EI\,i_B = \tfrac{1}{6} M_C L_2 + \tfrac{1}{3} M_B L_2 + \tfrac{1}{24} w_2 L_2^3 \quad . \quad . \quad (4)$$

By adding (3) and (4) i_B is eliminated, and we obtain an equation containing only the bending moments at three consecutive piers, the lengths of the two spans considered and the intensities of the load upon them, which is—

$$\tfrac{1}{6} M_A L_1 + \tfrac{1}{6} M_C L_2 + \tfrac{1}{3} M_B (L_1 + L_2) + \tfrac{1}{24}(w_1 L_1^3 + w_2 L_2^3) = 0,$$

or
$$4 M_A L_1 + 4 M_C L_2 + 8 M_B (L_1 + L_2) + w_1 L_1^3 + w_2 L_2^3 = 0 \quad (5)$$

If $L_1 = L_2 = L$, and $w_1 = w_2 = w$,

$$2 M_A + 2 M_C + 8 M_B + w L^2 = 0 \quad . \quad . \quad (6)$$

Equations (5) and (6) are the relationship sought between the moments at three consecutive piers.

To show how this is utilised, suppose the bridge has two equal spans, uniformly loaded, with the girders continuous over the centre pier (Fig. 84). Then as the girder is simply supported at A and C—*i.e.* prevented from moving up or down there—

$$M_A = M_C = 0.$$

Substituting these values in (6), we have—

$$M_B = -\frac{w L^2}{8}.$$

Taking moments about B, $M_B = R_C \times L - \frac{w L^2}{2}$. This, it will be noticed, is the difference between the moment of the actual reaction at C and its amount if the girder were severed at B and there was no stress couple there. Substituting for M_B,

$$R_C = \tfrac{3}{8} w L.$$

Consequently

$R_A = \frac{3}{8} w L$, and therefore $R_B = 2 w L - \frac{3}{4} w L = \frac{5}{4} w L$.

It will be observed that the reactions at the ends are less than what they would be if the girder were severed at B; this is obviously the case, because the stress moment at B causes the right-hand portion B C to tend to lift the left-hand portion in the clockwise sense; and since the action of the left-hand half on the right-hand half is equal and opposite, it tends to lift the right half in the contra-clockwise sense. Thus the reactions at A and C are diminished compared with their values if the girder is severed at B. If we imagine, instead of the uniform load $w L$ on B C, a load $\frac{w L}{2}$ at B and at C, the downward load $\frac{w L}{2}$ at C is only counteracted by the reaction $\frac{3}{8} w L$, the difference between these multiplied by L being equal to the moment of stress at B, which lifts the end C till the reaction is reduced from $\frac{w L}{2}$ to $\frac{3}{8} w L$, which, of course, necessitates the moment of stress at C being $\frac{w L^2}{8}$ (the action of the left portion on the right). This moment will necessarily be proportional to the distance of the section considered from C; if, therefore, we plot an ordinate $\frac{w L^2}{8}$ at B and join its extremity to A and C (Fig. 84), we have the diagram due to the moment at pier B. If we now plot on A B, B C, the parabolas for the bending moment diagram of the uniform

load on these two spans as if the pier moment were not present, the differences of the ordinates of these parabolas and those of the triangle A D C—*i.e.* the ordinates to the parabolas measured from the straight lines A D, D C, as a base—give the actual bending moment at each section. If these ordinates are plotted to a straight line *a c* as base, we obtain the diagram of resultant

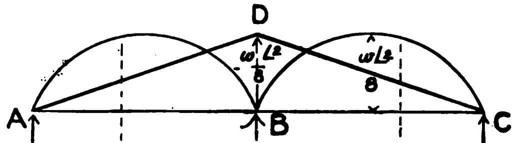


FIG. 84.

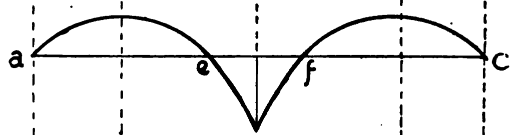


FIG. 85.

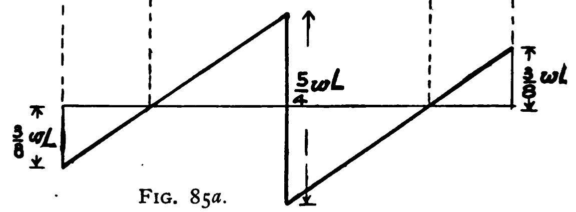


FIG. 85*a*.

bending moment as in Fig. 85. The shearing-force diagram is given in Fig. 85*a*, and since the shearing force is the differential coefficient of the bending moment at any section, it will be observed that the bending moment is a maximum when the shearing force equals zero.

Next suppose the bridge has three equal spans (Fig. 86), and is continuous over the two intermediate piers, the spans being uniformly loaded as before. As the ends A and D are simply prevented from moving vertically, $M_A = M_D = 0$, and applying equation (6) to the first three piers and the second three piers consecutively, we have—

$$8\,M_B + 2\,M_C + w\,L^2 = 0,$$

and

$$2\,M_B + 8\,M_C + w\,L^2 = 0.$$

Subtracting four times the second equation from the first,

$$2\,M_C - 32\,M_C + w\,L^2 - 4\,w\,L^2 = 0,$$

$$30\,M_C = -3\,w\,L^2,$$

$$\therefore\ M_C = -\frac{w\,L^2}{10} = M_B \text{ from symmetry.}$$

Taking moments about B to the left,

$$M_B = R_{AB}\,L - \frac{w\,L^2}{2} = -\frac{w\,L^2}{10},$$

$$\therefore\ R_A = R_{AB} = w\,L\,(\tfrac{1}{2} - \tfrac{1}{10}) = \tfrac{2}{5}\,w\,L = R_D,$$

$$\therefore\ R_B = R_C = \tfrac{1}{2}\,(3\,w\,L - \tfrac{4}{5}\,w\,L) = \tfrac{11}{10}\,w\,L.$$

It will be noticed in this case that the end reactions are nearer to their value when the girder is severed at B and C than the end reactions in the last case. The same remark also applies to the reactions at the intermediate piers; as the number of spans increases this is more and more the case.

If, as before, the bending moments at B and C be plotted (Fig. 86), and their extremities joined to each other and to A and

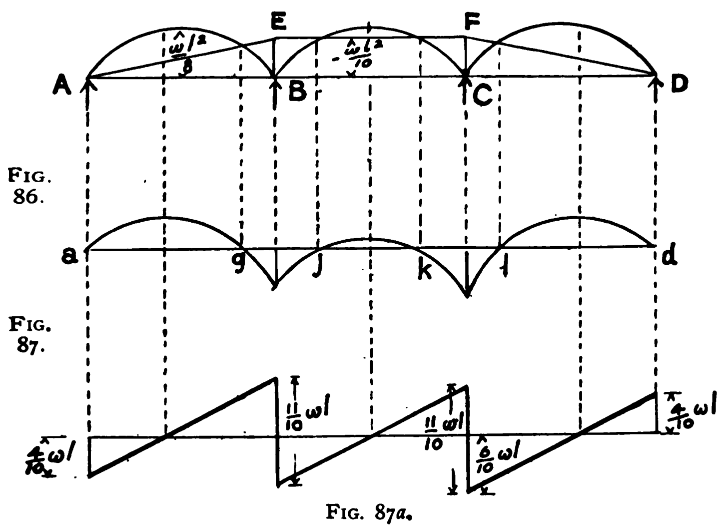


FIG. 87a.

D, the broken line A E F D represents the bending moment due to the pier moments. If the diagrams for a uniform load on each of the spans considered as separate be now drawn, the ordinates of these parabolas referred to the broken line A E F D as base represents the resultant bending moment diagram at each section. These ordinates may be plotted as in Fig. 87 relative to a straight base line $a\,d$, when the resultant bending

moment diagram is obtained. The shearing-force diagram is given in Fig. 87*a*, and it will be observed as before that the shearing force vanishes at the sections where the bending moment is a maximum.

It is obvious from Figs. 85 and 87 that the average value of the resultant bending moment at each section is considerably less than in the case of a series of separate beams of the same spans carrying the same load. Also the maximum bending moment occurs near the piers, in the position where an increased weight of the girders would affect the bending moment least. The necessary cross-section of the flanges may, apparently, therefore be considerably less in the case of the continuous girder than when it consists of independent spans; but the fact that the variations of stress in the members of the continuous girder are much greater than in the case of independent spans cause a higher factor of safety to be necessary in the former case, which partly neutralises the advantage of the reduced value of the maximum stress. Very great care has, however, to be taken in the erection to obtain this reduced average bending moment, and it is clear that if the spans are erected separately and then riveted together, it is necessary to lift the free end of a span whilst connecting the other end by an amount equal to its calculated deflection at the free end under its own weight, so as to put the top flange in tension over the piers.

It will also be obvious that a very small alteration in the level of a pier, either by settlement, or contraction or expansion of the pier, may have a very material effect on the resulting stresses in the girder. This point will be reverted to later.

Next find the bending moments and shearing forces for a girder of two spans continuous over the middle support, but with one span only (A B, Fig. 88) uniformly loaded and the other free from load. From equation (5) we have

$$4\,M_A + 16\,M_B + 4\,M_C + w\,L^2 = 0,$$

and $$M_A = M_C = 0, \therefore M_B = -\frac{w\,L^2}{16}.$$

Taking moments about B,

$$R_{AB} \times L - \frac{w\,L^2}{2} = M_B = -\frac{w\,L^2}{16},$$

$$\therefore R_{AB} = \tfrac{7}{16}\,w\,L, \quad \therefore R_{BA} = \tfrac{9}{16}\,w\,L.$$

Now $$R_{CB} = \frac{M_B}{L} = -\frac{w\,L}{16};$$

ιere is no other load on B C but the reactions,

$$R_{BC} = \frac{w\,L}{16};$$

$$\therefore\ R_B = R_{BA} + R_{BC} = (\tfrac{9}{16} + \tfrac{1}{16})\,w\,L = \tfrac{10}{16}\,w\,L.$$

' plotting in Fig. 88 first the triangular pier moment dia- and superimposing that for the uniform load on one span

FIG. 88.

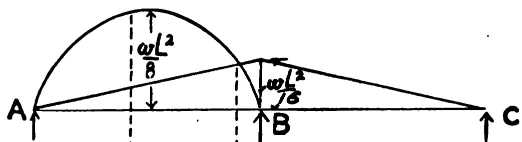
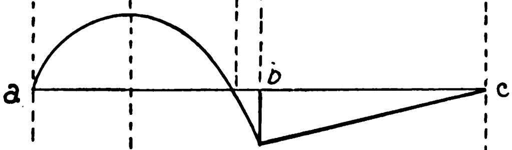


FIG. 88*a*.



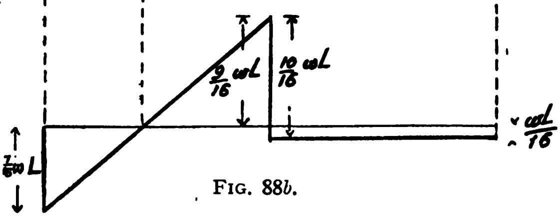


FIG. 88*b*.

ιnd measuring the ordinates relatively to the broken line of the :ormer, the resultant bending moment at each section is obtained; :his is plotted relatively to a straight base line *a c* in Fig. 88*a*. Fig. 88*b* is the shearing-force diagram plotted from the above results, the ordinate being of course zero when the bending moment is a maximum.

In a girder of three equal spans continuous over the two intermediate piers, find the bending-moment and shearing-force diagrams (1) when the centre span is uniformly loaded and the side spans unloaded; (2) when one side span is uniformly loaded and the other two spans unloaded.

(1) *The Centre Span Loaded*

From equation (5)

$$4\,M_A + 16\,M_B + 4\,M_C + w\,L^2 = 0,$$
$$4\,M_B + 16\,M_C + 4\,M_D + w\,L^2 = 0.$$

Since the extreme ends are simply prevented from moving vertically, $M_A = M_D = 0$, and from symmetry $M_B = M_C$.

Adding the two equations we get

$$20\,M_B + 20\,M_C + 2\,w\,L^2 = 0,$$

or

$$40\,M_B = -\,2\,w\,L^2,$$
$$\therefore\ M_B = M_C = -\frac{w\,L^2}{20}.$$

Taking moments about B,

$$R_A \times L = M_B = -\frac{w\,L^2}{20}, \quad \therefore\ R_D = R_A = -\frac{w\,L}{20},$$
$$\therefore\ R_B + R_C = w\,L - R_A - R_D = w\,L + \frac{w\,L}{10} = \frac{11}{10}\,w\,L,$$
$$\therefore\ R_B = R_C = \frac{11}{20}\,w\,L.$$

In Fig. 89, first the straight line diagram of the pier moment

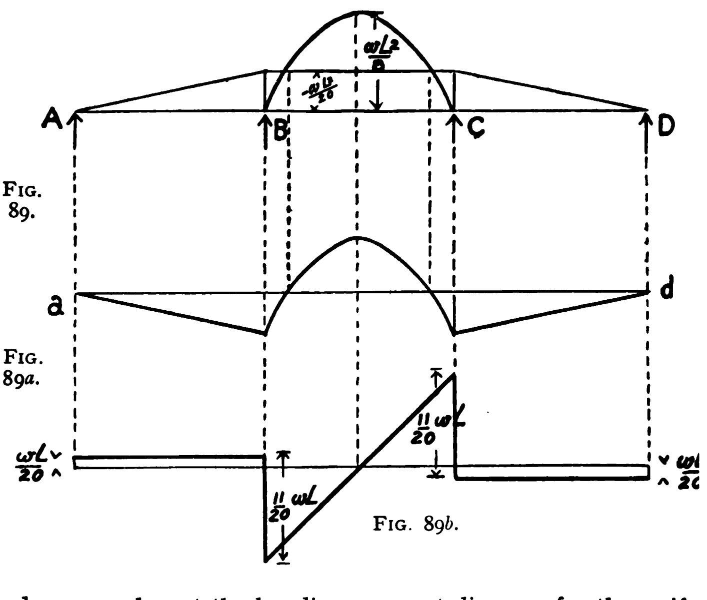


FIG. 89.

FIG. 89*a*.

FIG. 89*b*.

is drawn, and next the bending-moment diagram for the unifo load on B C is superimposed. The resultant diagram of bendi oments—*i.e.* the ordinates relatively to the broken line p

moment diagram—is plotted in Fig. 89*a* to the straight line *a d* as base. Fig. 89*b* is the shearing-force diagram.

(2) *One End Span Loaded*

From equation (5) $4\,M_A + 16\,M_B + 4\,M_C + w\,L^2 = 0$,
and
$$4\,M_B + 16\,M_C + 4\,M_D = 0.$$
Since the ends A and D are simply prevented from moving vertically, $M_A = M_D = 0$,

$$\therefore \text{ from the second equation } M_C = -\frac{M_B}{4}.$$

Substituting this value for M_C in the first equation—

$$16\,M_B - M_B + w\,L^2 = 0, \text{ or } M_B = -\frac{w\,L^2}{15};$$

$$\therefore M_C = \frac{w\,L^2}{60}.$$

Now $$R_D \times L = M_C = \frac{w\,L^2}{60}, \quad \therefore R_D = \frac{w\,L}{60},$$
and $$M_B = R_D \times 2\,L + R_C \times L,$$
because R_D and R_C are the only external forces to the right of B;

$$\therefore M_B = 2\,M_C + R_C \times L,$$

$$\therefore R_C\,L = M_B - 2\,M_C = \tfrac{3}{2}\,M_B = -\frac{w\,L^2}{10}, \text{ or } R_C = -\frac{w\,L}{10}.$$

Taking moments about B to the left,

$$R_{AB} \times L - \frac{w\,L^2}{2} = M_B = -\frac{w\,L^2}{15},$$

$$\therefore R_A = R_{AB} = \frac{13}{30}\,w\,L.$$

Now $R_A + R_B + R_C + R_D = w\,L$.
Substituting for R_A R_C and R_D, we find that

$$R_B = w\,L - \frac{13}{30}\,w\,L + \frac{w\,L}{10} - \frac{w\,L}{60}, \text{ or } R_B = \frac{39}{60}\,w\,L.$$

In Fig. 90 the pier moments diagram is first plotted and the bending-moment diagram for the uniform load on A B is superimposed, then the ordinates of the bending-moment diagram for the loads, minus the corresponding ordinate of the pier moment diagram, is the resultant bending moment at the section considered. These are plotted to a straight line base *a d* in Fig. 90*a*.

The shearing-force diagram (Fig. 90*b*) can be easily plotted from the values of the reactions which have just been calculated.

Comparing the two diagrams of bending moments, Fig. 89a and Fig. 90a, it is seen that when the live load covers one end span the maximum positive bending moment due to it occurs in

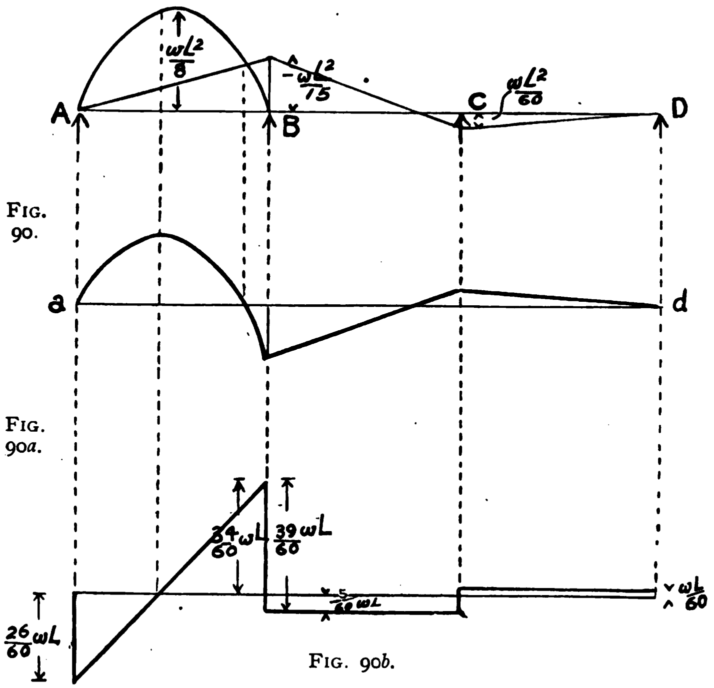


FIG. 90.

FIG. 90*a*.

FIG. 90*b*.

that span, and the maximum negative bending moment at the intermediate pier at the end of that span, the bending moment for the greater part of the centre span is negative, whilst that in the other end span is positive. The maximum positive shear then occurs at the intermediate pier at the end of the loaded span, and the maximum negative shear at the other end of that span.

'When the load covers the centre span only, the central portion of that span is subject to positive bending moments, the maximum negative bending moments being at the intermediate piers at either end of this span, and the maximum positive and negative shears occur at these piers.

A much more accurate estimation of the maximum bending moments and shearing forces due to the live load can, however,

be arrived at by considering the effect of a single concentrated load, and then examining what positions of such loads give positive and negative values of the bending moments and shearing forces respectively; then by taking only the portions loaded which give moments or shears of the same sign at the section considered, the maximum values can be ascertained.

For this purpose it is necessary to examine the modified form of the theorem of three moments for concentrated loads.

Theorem of Three Moments for Concentrated Loads

We will first consider a single load Q to act at X (Fig. 91) distant ξ from A. The difference between this case and that for

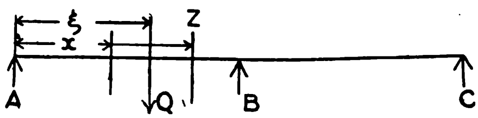


FIG. 91.

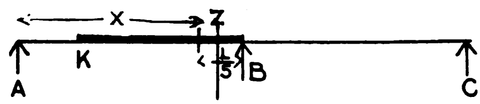


FIG. 92.

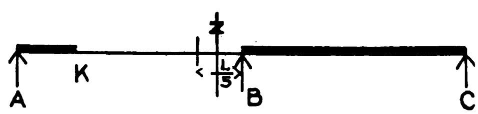


FIG. 92*a*.

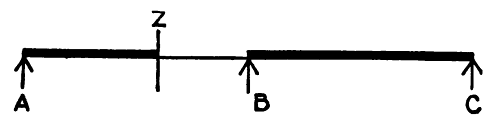


FIG. 93.

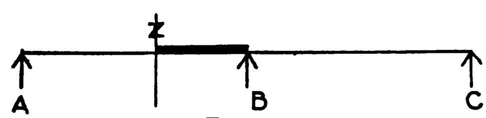


FIG. 94.

the uniform load is that the expression for the bending moment is discontinuous at the load. It is necessary, therefore, to integrate the changes of inclination from A to X for the one

expression for the bending moment, and from X to B for the other expression for it. In order to get rid of the unknown value of the inclination of the centre line at A, the fact that no deflection takes place at the piers has to be introduced, and the deflection at X has to be found for both expressions for the bending moment and the two values equated. Let M be the bending moment at any point z distant x from A, and use the same notation as before, the length of A B being L_1. Taking A for the origin and the axis of x in the direction A B—

between A and X, $M = M_A + R_{AB}\,x$;

between X and B, $M = M_A + R_{AB}\,x - Q\,(x - \xi)$.

For a section between A and X, the total change of inclination from A to the section distant x from A is found by integrating the expression $\dfrac{1}{EI}\displaystyle\int_0^x M\,dx$, where M is the expression for the bending moment between A and X; that is:—

$$EI\,(i - i_A) = \int_0^x (M_A + R_{AB} \times x)\,dx = M_A\,x + R_{AB} \times \frac{x^2}{2},$$

$$\therefore\ EI\,i = EI\,i_A + M_A\,x + R_{AB} \times \frac{x^2}{2} \quad . \quad . \quad . \quad (1)$$

At X therefore $EI\,(i_X - i_A) = M_A\,\xi + R_{AB} \times \dfrac{\xi^2}{2}$,

$$\text{or} \qquad EI\,i_A = E\ \ i_X - M_A\,\xi - R_{AB} \times \frac{\xi^2}{2} \quad . \quad . \quad . \quad (2)$$

To obtain a second expression involving i_A, find the difference in the deflection between A and X, remembering that the deflection at A = 0.

Integrating (1) between A and X, we have for the increase of the deflection from A to X—

$$EI\,(u_X - u_A) = EI\int_0^{\xi} i\,dx = EI\,i_A\,\xi + M_A\frac{\xi^2}{2} + R_{AB}\frac{\xi^3}{6};$$

or, since $u_A = 0$,

$$EI\,(u_X - i_A\,\xi) = M_A\frac{\xi^2}{2} + R_{AB}\frac{\xi^3}{6} \quad . \quad . \quad . \quad (3)$$

Next, the increase of inclination between X and a section distant x from A between X and B

$$= \int_{\xi}^{x} M\,dx,$$

where M is the expression for the bending moment between X and B; that is:—

$$\mathrm{EI}\,(i - i_{\mathrm{X}}) = \int_{\xi}^{x} \{\mathrm{M_A} + \mathrm{R_{AB}} \times x - \mathrm{Q}\,(x - \xi)\}\, d\,x,$$

Or $$\mathrm{EI}\,i = \mathrm{M_A}\,x + \mathrm{R_{AB}}\frac{x^2}{2} - \frac{\mathrm{Q}\,(x-\xi)^2}{2} + \mathrm{EI}\,i_{\mathrm{X}} - \mathrm{M_A}\,\xi - \mathrm{R_{AB}}\frac{\xi^2}{2}.$$

Substituting for the last three terms from (2),

$$\mathrm{EI}\,i = \mathrm{M_A}\,x + \mathrm{R_{AB}}\frac{x^2}{2} - \mathrm{Q}\frac{(x-\xi)^2}{2} + \mathrm{EI}\,i_{\mathrm{A}}.$$

If $x = \mathrm{L_1}$,

$$\mathrm{EI}\,i_{\mathrm{B}} = \mathrm{M_A}\,\mathrm{L_1} + \mathrm{R_{AB}}\frac{\mathrm{L_1}^2}{2} - \mathrm{Q}\frac{(\mathrm{L_1}-\xi)^2}{2} + \mathrm{EI}\,i_{\mathrm{A}} \quad . \quad . \quad (4)$$

To obtain a second expression involving i_{A}, find the difference in the deflection between X and B by integrating the above expression for i between X and B, remembering that the deflection at B = 0, we have—

$$\mathrm{EI}\,(u_{\mathrm{B}} - u_{\mathrm{X}}) = \mathrm{EI}\int_{\xi}^{\mathrm{L_1}} i\,d\,x = \mathrm{M_A}\left(\frac{\mathrm{L_1}^2}{2} - \frac{\xi^2}{2}\right) + \mathrm{R_{AB}} \times \left(\frac{\mathrm{L_1}^3}{6} - \frac{\xi^3}{6}\right)$$
$$- \frac{\mathrm{Q}\,(\mathrm{L_1}-\xi)^3}{6} + \mathrm{EI}\,i_{\mathrm{A}}\,(\mathrm{L_1} - \xi),$$

making $u_{\mathrm{B}} = 0$, and bringing the last term on the right over to the left,

$$\mathrm{EI}\,(u_{\mathrm{X}} - i_{\mathrm{A}}\,\xi) + \mathrm{EI}\,i_{\mathrm{A}}\,\mathrm{L_1} = -\,\mathrm{M_A}\left(\frac{\mathrm{L_1}^2}{2} - \frac{\xi^2}{2}\right) - \mathrm{R_{AB}} \times \left(\frac{\mathrm{L_1}^3}{6} - \frac{\xi^3}{6}\right)$$
$$+ \frac{\mathrm{Q}\,(\mathrm{L_1}-\xi)^3}{6}.$$

Substituting for the first term on the left from (3),

$$\mathrm{M_A}\frac{\xi^2}{2} + \mathrm{R_{AB}} \times \frac{\xi^3}{6} + \mathrm{EI}\,i_{\mathrm{A}}\,\mathrm{L_1} = -\,\mathrm{M_A}\left(\frac{\mathrm{L_1}^2}{2} - \frac{\xi^2}{2}\right) - \mathrm{R_{AB}}$$
$$\times \left(\frac{\mathrm{L_1}^3}{6} - \frac{\xi^3}{6}\right) + \mathrm{Q}\,\frac{(\mathrm{L_1}-\xi)^3}{6},$$

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$$\mathrm{EI}\,i_{\mathrm{A}}\,\mathrm{L_1} = -\,\mathrm{M_A}\,\frac{\mathrm{L_1}^2}{2} - \mathrm{R_{AB}}\frac{\mathrm{L_1}^3}{6} + \frac{\mathrm{Q}\,(\mathrm{L_1}-\xi)^3}{6}.$$

Multiplying (4) by $\mathrm{L_1}$ and substituting in it the value just found for $\mathrm{EI}\,i_{\mathrm{A}}\,\mathrm{L_1}$, we get

$$\mathrm{EI}\,i_{\mathrm{B}}\,\mathrm{L_1} = \mathrm{M_A}\,\mathrm{L_1}^2 + \mathrm{R_{AB}} \times \frac{\mathrm{L_1}^3}{2} - \frac{\mathrm{Q}\,(\mathrm{L_1}-\xi)^2}{2} \times \mathrm{L_1} - \mathrm{M_A}\,\frac{\mathrm{L_1}^2}{2}$$
$$- \mathrm{R_{AB}}\frac{\mathrm{L_1}^3}{6} + \frac{\mathrm{Q}\,(\mathrm{L_1}-\xi)^3}{6}.$$

Collecting the terms,

$$\mathrm{E\,I}\,i_{\mathrm{B}}\,\mathrm{L}_1 = \mathrm{M}_{\mathrm{A}}\frac{\mathrm{L}_1^{\,2}}{2} + \mathrm{R}_{\mathrm{AB}}\frac{\mathrm{L}_1^{\,3}}{3} - \mathrm{Q}\,(\mathrm{L}_1 - \xi)^2\left(\frac{\mathrm{L}_1}{2} - \frac{\mathrm{L}_1 - \xi}{6}\right)$$

$$= \mathrm{M}_{\mathrm{A}}\frac{\mathrm{L}_1^{\,2}}{2} + \mathrm{R}_{\mathrm{AB}}\frac{\mathrm{L}_1^{\,3}}{3} - \mathrm{Q}\,\frac{(\mathrm{L}_1 - \xi)^2\,(2\,\mathrm{L}_1 + \xi)}{6}.$$

Now taking moments about B—

$$\mathrm{R}_{\mathrm{AB}} \times \mathrm{L}_1 = \mathrm{M}_{\mathrm{B}} - \mathrm{M}_{\mathrm{A}} + \mathrm{Q}\,(\mathrm{L}_1 - \xi) \quad . \quad . \quad . \quad (5)$$

Substituting this value for R_{AB} in the last equation,

$$\mathrm{E\,I}\,i_{\mathrm{B}}\,\mathrm{L}_1 = \mathrm{M}_{\mathrm{A}}\frac{\mathrm{L}_1^{\,2}}{6} + \mathrm{M}_{\mathrm{B}}\frac{\mathrm{L}_1^{\,2}}{3} - \mathrm{Q}\,\frac{(\mathrm{L}_1 - \xi)^2\,(2\,\mathrm{L}_1 + \xi)}{6} + \frac{\mathrm{Q}\,\mathrm{L}_1^{\,2}\,(\mathrm{L}_1 - \xi)}{3},$$

$$\therefore \qquad \mathrm{E\,I}\,i_{\mathrm{B}} = \mathrm{M}_{\mathrm{A}}\frac{\mathrm{L}_1}{6} + \mathrm{M}_{\mathrm{B}}\frac{\mathrm{L}_1}{3} + \frac{\mathrm{Q}\,\xi\,(\mathrm{L}_1^{\,2} - \xi^2)}{6\,\mathrm{L}_1}.$$

If there are a number of separate loads, each load increases the inclination and the moments M_{A} and M_{B}; thus a similar expression to the above would be obtained for each load. The sum of the terms corresponding to i_{B} would be the resultant change of inclination.

The sum of the values of M_{A} in the first term on the right would be the resultant moment at A, and similarly for M_{B} in the second term.

Therefore if i_{B} is the total inclination due to all the loads on the span A B, and M_{A} and M_{B} are the total moments due to these loads, we should have—

$$\mathrm{E\,I}\,i_{\mathrm{B}} = \mathrm{M}_{\mathrm{A}}\frac{\mathrm{L}_1}{6} + \mathrm{M}_{\mathrm{B}}\frac{\mathrm{L}_1}{3} + \Sigma\,\frac{\mathrm{Q}\,\xi\,(\mathrm{L}_1^{\,2} - \xi^2)}{6\,\mathrm{L}_1} \quad . \quad . \quad (6)$$

where the Σ denotes the summation of the last term for each load Q, ξ being the distance of its point of application from A.

If now, as in the case of the uniformly distributed load, we take C for the origin and the axis of x in the direction C B, and let Q^1 be any load on the span B C at the distance η from C, we have the tangent of the supplement of the angle in (6). L_1 becomes L_2, the length of the span B C, and A becomes C, therefore

$$-\mathrm{E\,I}\,i_{\mathrm{B}} = \mathrm{M}_{\mathrm{C}}\frac{\mathrm{L}_2}{6} + \mathrm{M}_{\mathrm{B}}\frac{\mathrm{L}_2}{3} + \Sigma\,\frac{\mathrm{Q}^1\,\eta\,(\mathrm{L}_2^{\,2} - \eta^2)}{6\,\mathrm{L}_2} \quad . \quad . \quad (7)$$

Adding (6) and (7) and multiplying by 6,

$$\mathrm{M}_{\mathrm{A}}\,\mathrm{L}_1 + 2\,\mathrm{M}_{\mathrm{B}}\,(\mathrm{L}_1 + \mathrm{L}_2) + \mathrm{M}_{\mathrm{C}}\,\mathrm{L}_2 + \Sigma\,\frac{\mathrm{Q}\,\xi\,(\mathrm{L}_1^{\,2} - \xi^2)}{\mathrm{L}_1}$$

$$+ \Sigma\,\frac{\mathrm{Q}^1\,\eta\,(\mathrm{L}_2^{\,2} - \eta^2)}{\mathrm{L}_2} = 0 \quad . \quad . \quad . \quad . \quad (8)$$

which is the theorem of three moments for concentrated loads.

As an application of this, take a girder with two equal spans

continuous over the central pier with a single load Q on the span A B distant ξ from A, and find the bending moment at a point z distant x from A.

In this case $M_A = M_C = 0$, therefore from (8)—

$$4 M_B L + Q \xi \frac{(L^2 - \xi^2)}{L} = 0,$$

$$\text{or } M_B = - \frac{Q}{4 L^2} \xi (L^2 - \xi^2).$$

From (5) $\quad R_{AB} \times L = M_B + Q (L - \xi),$

$$\therefore R_A = R_{AB} = - \frac{Q}{4 L^3} \xi (L^2 - \xi^2) + \frac{Q (L - \xi)}{L}$$

$$\text{and } R_C = R_{CB} = \frac{M_B}{L} = - \frac{Q}{4 L^3} \xi (L^2 - \xi^2)$$

$$\therefore R_B = Q - R_A - R_C = Q \frac{\xi}{L} + \frac{Q}{2 L^3} \xi (L^2 - \xi^2).$$

It will be noticed that R_B is always positive and R_A is always positive, because the second term in the expression for R_A is greater than the first for $1 > \frac{\xi (L + \xi)}{4 L^2}$. R_C, the reaction at the extremity of the span free from load, is always negative. Between A and the load Q, the bending moment $= R_{AB}\, x$, and is always positive; between the load Q and B the bending moment equals—

$$\begin{aligned} M &= R_{AB}\, x - Q (x - \xi), \\ &= \left\{ \frac{Q (L - \xi)}{L} - \frac{Q}{4 L^3} \xi (L^2 - \xi^2) \right\} x - Q (x - \xi), \\ &= Q \xi \left\{ 1 - \frac{x}{4 L^3} (4 L^2 + L^2 - \xi^2) \right\}, \\ &= Q \xi \left\{ 1 - \frac{x}{4 L^3} (5 L^2 - \xi^2) \right\}, \\ M &= 0 \text{ if } x = \frac{4 L^3}{5 L^2 - \xi^2}. \end{aligned}$$

Position of Loads for Maximum Bending Moment at a Section

M is positive if $\quad x (5 L^2 - \xi^2) < 4 L^3;$

that is, if $\quad \xi^2 > L^2 \left(5 - \frac{4 L}{x}\right).$

ξ^2 is obviously greater than the expression on the right if $5 - \frac{4 L}{x} = 0$, *i.e.* if $x = \frac{4}{5} L$; it is also greater for all smaller

values of x than this, because as x becomes smaller the negative term in the expression is increased. But if $x > \frac{4}{5}$ L, the bending moment will not be positive until ξ is large enough to be $> \mathrm{L}\sqrt{5 - 4\frac{\mathrm{L}}{x}}$, *i.e.* it will be negative if ξ is $< \mathrm{L}\sqrt{5 - 4\frac{\mathrm{L}}{x}}$. It will also always be negative for all loads on the span B C, because R_{A} is negative for all loads on B C. Therefore for a section in the four-fifths of the span from A the positive bending moment at Z is a maximum if A B is covered by the load, and the maximum negative bending moment at Z occurs when the span B C only is covered. But if $x > \frac{4}{5}$ L, the bending moment at Z is a maximum, as shown in Fig. 92, if there is no load on B C and no load from A to a point K at a distance equal to $\mathrm{L}\sqrt{5 - 4\frac{\mathrm{L}}{x}}$ from A, but with the load covering the span from that point to B. The maximum negative bending moment at Z (Fig. 92*a*) occurs in this case if the length A K and the span B C are loaded.

Position of Loads to give Maximum Shearing Force at any Section

It has already been shown that for a load between A and B the reactions at A and B are positive, and that at C negative; therefore for a load to the left of any section Z distant x from A in A B the resultant force to the left of Z is downwards, because the load is greater than its reaction at A, therefore the shearing force at Z is positive; for a load between Z and B the resultant force to the left of Z is the reaction at A and therefore upwards, which gives a negative shearing force at Z; for a load on the span B C the reaction at A is negative, therefore the resultant force to the left of Z is downwards, and the shearing force at Z positive. The maximum positive shear will therefore occur at Z (Fig. 93), when the portion A Z of the span A B is loaded and the span B C is also loaded. The maximum negative shear occurs at Z (Fig. 94), when the portion Z B of the span A B is loaded and the rest of the girder A C is free from loads.

CANTILEVER BRIDGES

In the case of a continuous girder of two spans each of length L, carrying a uniform load, we have seen that if the piers are

t the same level the centre pier takes five-fourths of the span-oad. If the centre pier were to sink an amount equal to the de-lection of the girder with span A C, when carrying the whole load, upposing it to be strong enough to do so, there would then be o pressure on the central pier, and the whole load would come n the end piers. The central pier would have to sink a dis-ance u given by the formula in Chapter IX., page 168—viz. :—

$$u = \frac{5}{24} \times \frac{(2\,L)^2 \times f_c}{E \times D}, \text{ where } f_c = \frac{w\,(2\,L)^2}{8\,A \times D},$$

$$\therefore\ u = \tfrac{5}{12}\,\frac{w\,L^4}{E \times A \times D^2}.$$

On the other hand, if the central pier expands in length so s to lift the girder A C with its load off its end supports, it would ake the whole load and the end supports nothing. The amount t would have to rise to effect this is given on page 170, and is u', where $u' = \tfrac{2}{3}\,\frac{L^2}{E \times D}\,(f_B - f_1)$.

In this case $f_B = \frac{w\,L^2}{2\,A \times D}$ and $f_1 = \frac{w\,L^2}{8\,A \times D}$,

$$\therefore\ u' = \tfrac{2}{3}\,\frac{L^2}{E \times D} \times \tfrac{3}{8}\,\frac{w\,L^2}{A \times D} = \tfrac{1}{4}\,\frac{w\,L^4}{E \times A \times D^2}.$$

Thus it will be seen that if the central pier is $\tfrac{5}{12}\,\frac{w\,L^4}{E \times A \times D^2}$ below the line of the end piers, there would be no pressure upon t, and if it is $\tfrac{1}{4}\,\frac{w\,L^4}{E \times A \times D^2}$ above that line it will carry the whole load $2\,w\,L$. That is to say, a variation in height of the centre pier equal to $(\tfrac{5}{12} + \tfrac{1}{4})\,\frac{w\,L^4}{E \times A \times D^2} = \tfrac{2}{3}\,\frac{w\,L^4}{E \times A \times D^2}$ in-creases the pressure upon it from 0 to $2\,w\,L$, and for any inter-mediate height, since we are considering an elastic body, the pressure upon it is the same fraction of the total load that the distance above the position of no load is of $\tfrac{2}{3}\,\frac{w\,L^4}{E \times A \times D^2}$.

Thus, when the three piers are level the centre one is $\tfrac{5}{12} \times \frac{w\,L^4}{E \times A \times D^2}$ above the position for no load on it ; the pressure on t therefore is $(\tfrac{5}{12} \div \tfrac{2}{3}) \times 2\,w\,L = \tfrac{5}{4}\,w\,L$, which, of course, is the same result as previously obtained.

It will be obvious from the above considerations that alterations in the level of any of the piers supporting a continuous

girder may produce important changes in the relative amount of reaction upon the piers, but if in Fig. 85 the centre portion of the girder length *e f* forms a cantilever supported at its middle point and the remaining portions of the span, length *a e* and *c f* respectively, are hinged to it at *e* and *f*, or simply supported on the ends of the cantilever and on the end piers, any moderate alteration in the level of the piers can no longer materially affect the stresses in the girder, because it can now bend at *e* and *f* to accommodate the altered level of the piers.

Again, an examination of Fig. 87 shows that if the girder be severed at the points *g*, *j*, *k*, and *l* in its length, where the bending moment becomes zero, and the portion *g j* and *k l* form cantilevers supported at B and C, and the lengths *a g*, *j k*, and *l d* simply rest on the extremities of the cantilevers, or are connected to them by hinges, so that no bending moment can be transmitted through them, as indicated in Fig. 95, then it

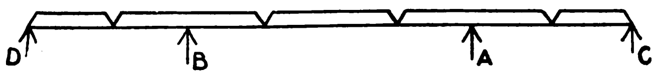

FIG. 95.

will be seen that any alteration in the height of the piers will have little effect on the stresses in the girder, as bending can take place at the points of connection without introducing stresses in the members of the girder. The effect, however, of so many joints in a bridge would be to cause a lack of rigidity. To avoid this, the joints in the end spans may be omitted without affecting the advantage derived by introducing discontinuity; for in this case (Fig. 96) we have two shore spans with projecting cantilever arms, each forming a single girder supported at one end and at a point distant from the other end by the length of the cantilever arm, with a girder between them carried on the extremities of these arms. This is a very convenient form for enabling the centre span to be built out from the shore spans after their erection, without the aid of staging. Such an arrangement is indicated in Fig. 96, where C E and D F are the shore spans and cantilever arms, supported at C and A and D and B respectively. The central girder is supported on the ends of the cantilever arms A E and B F. Such a structure is statically determinate, a load on the shore span A (or D B) causes a positive reaction at C and A (or D and

a load anywhere between A and B will cause a positive tion at A and B but a negative reaction at C and D, con-.ently the girder must be prevented from lifting at these .ts. Instead of single pier supports at A and B, it is very mon to provide a double support as indicated in Fig. 97 . A′ and B B′. If the panel lengths A A′ and B B′ are diagon-braced so as to transmit shear, C E and D F become continuous

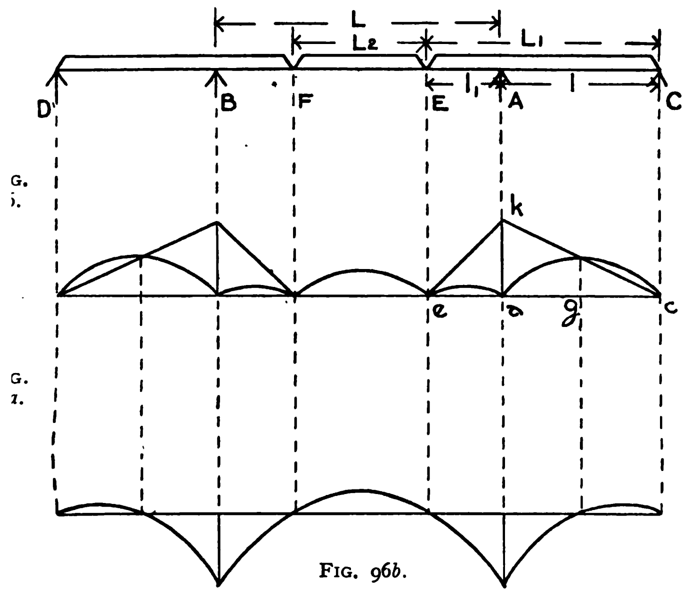

FIG. 96*b*.

lers, and should be treated as such; but to avoid this result ;e panels may be braced horizontally so as to stiffen the tical struts at A and A′ without transmitting shear between :n. In the latter case, as the shearing force in the panel ero, the bending moment at A and A′ is the same.

A Single Support at A *and* B

First consider a uniform dead load of intensity w covering bridge. As the central span E F of a length equal L_2 is ply supported at the ends, one-half its load $\frac{w\,L_2}{2}$ will be trans-ted to E and the other half to F.

The bending moment at A, M_A, is therefore the moment of $\frac{w L_2}{2}$, and that of the uniform load on the cantilever arm A E whose length is l_1. That is—

$$M = -\frac{w L_2}{2} \times l_1 - \frac{w l_1^2}{2} = -\frac{w l_1}{2}(l_1 + L_2).$$

If an ordinate $a\,k$ (Fig. 96*a*) be drawn equal to this moment, and its extremity k be joined to c and e, and the parabolas be drawn on $a\,c$ and $a\,e$ due to a uniform load equal w tons per linear foot, the ordinates of the parabolas, measured from the broken pier moment line $c\,k\,e$ as base, will give as before the resultant bending moment diagram. These ordinates are plotted relatively to a straight line base in Fig. 96*b*.

Taking moments about C,

$$R_A = \frac{w L_2}{2} \times \frac{L_1}{l} + \frac{w L_1^2}{2\,l} = \frac{w L_1}{2\,l}(L_1 + L_2);$$

$$\therefore\ R_C = w\left(L_1 + \frac{L_2}{2}\right) - R_A = w\left(L_1 + \frac{L_2}{2}\right) - \frac{w L_1}{2\,l}(L_1 + L_2).$$

This is negative if

$$L_1 + \frac{L_2}{2} < \frac{L_1}{2l}(L_1 \times L_2)$$

—*i.e.* if

$$l < \frac{L_1(L_1 + L_2)}{2\,L_1 + L_2}.$$

To see the effect of a live load it is best, as before, to take a single load Q, say, in different positions on the bridge.

It is clear that a load on D F has no effect on the corresponding girder C E.

A Load Q *on the Central Span distant* ξ *from* F

This causes a load at E, the extremity of the cantilever arm

$$A\,E = \frac{Q \times \xi}{L_2}.$$

Taking moments about C the reaction at A

$$= R_A = \frac{Q \times \xi}{L_2} \times \frac{L_1}{l},$$

and the reaction at C

$$= R_C = \frac{Q \times \xi}{L_2} - \frac{Q \times \xi}{L_2} \times \frac{L_1}{l} = -\frac{Q \times \xi}{L_2}\left(\frac{L_1 - l}{l}\right) = -\frac{Q \times \xi}{L_2} \times \frac{l_1}{l}.$$

A Load Q *on the Cantilever Arm* A E *distant x from* A

The reaction at $A = R_A = Q\,\frac{x+l}{l} = Q\left(1+\frac{x}{l}\right)$,

nd the reaction at $C = R_C = Q - R_A = -.\,Q\frac{x}{l}$.

A Load Q *on the Shore Span* A C *distant* x^1 *from* A

$$R_A = Q \times \frac{l-x^1}{l}, \text{ and } R_C = Q\frac{x^1}{l},$$

which are exactly the same as for a girder span A C supported ıt A and C. Thus a load in any position gives a positive reaction ıt A, loads to the left of A give a negative reaction at C, and oads to the right of A give a positive reaction at C.

At any section between A and C the maximum positive hear therefore occurs when the load extends from A to ıe section only, and the maximum negative shear when the ›ad extends from the section to C, and also covers the length F to the left of A. At any section between A and E the maxiıum positive shear occurs when the length between the section ıd F is loaded, and it is immaterial whether the length between ıe section and A is loaded, as the sum of the reactions at A nd C due to such load is equal to the load. Loads on A C ,ould not affect the shear in the cantilever arm, as the rections at A and C due to such a load are equal to it; it is herefore immaterial in this respect whether they are present r not.

For E F the shears are exactly the same as for a girder of that pan. The bending moment at A is always negative, and its ıaximum value occurs when the length from A to F is fully ›aded; in this case also a load on A C does not affect the value f the bending moment, as the moment at A due to any such ›ad is zero.

The bending moments in the shore span A C are positive :om *g* to *c*, Fig. 96*a*, and their maximum value of course occurs ,hen that span is fully loaded.

Double Suppotrs at A *and* B, when the panel between the two upports are arranged so as not to transmit shear (Fig. 97).

The bending moment diagram for a uniformly distributed ›ad will be as before, with a constant value between A and A^1 Fig. 97*a*), because the shear there is zero.

L

A load on the central span distant ξ from F transmits, as before, a load equal to $\frac{Q \times \xi}{L_2}$ to E, therefore, since no shear is transmitted from A to A^1, $R_A = \frac{Q \times \xi}{L_2}$.

Now the moment at A due to this load $= -\frac{Q \times \xi}{L_2} \times l_1$, which is also the moment at A^1, ∴ the reaction at C

$$= R_C = -\frac{Q \times \xi}{L_2} \times \frac{l_1}{l},$$

and the reaction at $A^1 = R_{A^1} = -R_C = \frac{Q \times \xi}{L_2} \times \frac{l_1}{l}$.

FIG. 97.

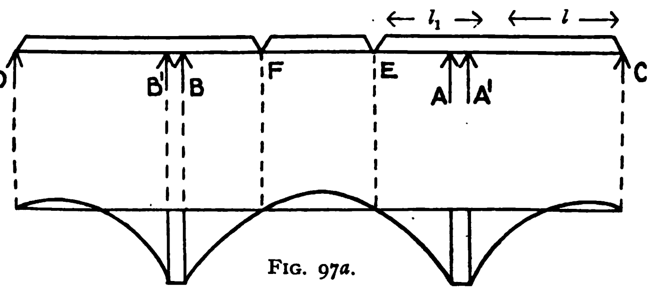

FIG. 97*a*.

It will be noticed that R_C has the same value as for the single support, and now $R_A + R_{A^1} = \frac{Q \times \xi}{L_2} \times \frac{l + l_1}{l} = \frac{Q \times \xi}{L_2} \times \frac{L_1}{l}$ equals the reaction at the single support.

A load Q *on the cantilever arm* A E *distant x from* A.

$R_A = Q$

$M_A = -Q \times x$, ∴ $M_{A^1} = -Q \times x$, ∴ $R_C = -\frac{Q \times x}{l}$,

and $R_{A^1} = \frac{Q \times x}{l}$.

As in the last case, R_C has the same value as for the single support, and now $R_A + R_{A^1} = Q\left(1 + \frac{x}{l}\right)$, which equals the reaction at the single support.

A load Q *on the shore span* A C, as before, causes positive reactions at A^1 and C, just as in the case of a girder of that span supported at the ends. The same remarks as to maximum bending moments and shearing stresses, as in the case of the single support, apply also in this case.

CHAPTER IX

DEFLECTION OF GIRDERS

THERE is an essential difference between the deflection of a girder with a continuous web and a braced girder.

In the case of a girder with continuous web there are, as has been shown, tensions and compressions acting on certain planes in the web, but corresponding to the tension on one side of the neutral axis there are compressions on the other side, therefore the elongations and contractions due to them are more or less balanced in the vertical direction. The shearing stress in the web will, however, cause a deflection, but as this is simply due to the sliding of each vertical section relatively to its neighbour, the total deflection due to shear is simply that caused by the vertical shear between the two sections considered, and the deflection at any one vertical section does not affect that at others. Generally speaking, the deflection due to the bending in any element of length causes a gradually increasing deflection in other sections as their distance from it increases, and therefore the deflection in an element of length is not only that due to its own bending, but also to the magnified effects of the bending of the elements of length of the beam on either side of it. This being the case, the deflection due to shear is very small compared with that due to bending, and may generally be neglected.

In the case of a braced girder there is no shearing stress, as shear, in the braces, but the shear is taken up by the tension or compression in the braces; these are consequently lengthened or shortened, as the case may be. This alteration in length of the braces affects the new shape taken up by each bay when subjected to the load, as is readily seen in Fig. 106. The originally vertical lines in a girder maintain a position which is radial to the curve of deflection, and therefore the deflection at any point may be measured from the tangent at another point of this curve—*e.g.*, in a girder supported at both ends and symmetrically loaded, the deflection of the ends upwards may

be measured from the horizontal tangent at the centre of the girder, or the upward deflection of the centre relative to the new position of the tangents at the ends may be found.

Deflection of Girder with Continuous Web

First consider the case of a girder with continuous web and divide it into a number of bay lengths, and consider the bay A B D C (Fig. 98), and let the deflection at any vertical section $y\,y^1$ (Fig. 99) be measured relatively to the tangent to the dotted deflection curve $y\,z$ at z, the centre of the vertical A C, which is normal to the tangent at z. With a positive bending moment curve the upper flange will be shortened, and the lower flange lengthened. In Fig. 98 let B b denote the shortening of the upper flange and D d the lengthening of the lower one under the load. With centres A and C draw arcs of circles with radii A b and C d respectively; as the deflection is so small compared with these radii, the circular arcs will be straight lines $b\,b^1$ and $d\,d^1$ perpendicular to A B and C D. Suppose that in the bay considered the shearing force is negative, so that the diagonal B C, if the girder were braced,

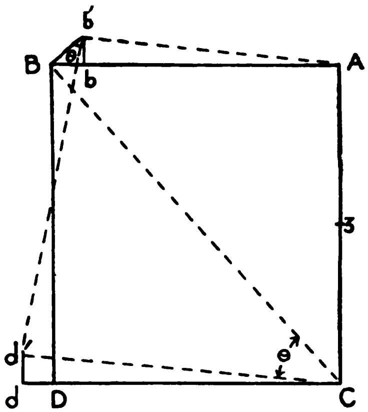

Fig. 98.

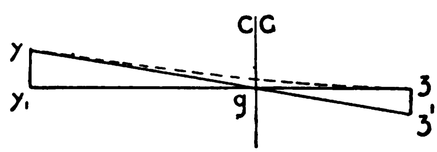

Fig. 99.

would be in tension. In the case of a girder with continuous web, the diagonal B C remains of constant length, therefore with centre C and radius C B describe a circle which intersects the circle centre A at b^1. Then C b^1 is the new position of the diagonal; and if with b^1 as centre an arc of a circle be

described with radius equal to the vertical B D to cut the circle centre C in d^1, d^1 will be the new position of D, therefore A b^1 d^1 C is the new position of the bay. With a continuous web the flanges will bend to a curve which may be taken to be a circle. Thus the upward deflection $b\,b^1$ relatively to A B will be the deflection of the tangent to the deflection curve at b^1 from that at A, and the tangent at b^1 will meet A B at the centre of its length (of course it must be borne in mind that B b is relatively insignificant in comparison with A B). Calling D the depth of the girder and b the bay length taken, the deflection $b\,b^1 = \overline{\text{B}\,b} \times \frac{b}{\text{D}}$, because the triangles B b^1 b and B C D are similar, their sides being mutually normal to each other. The effect of the displacement of B will be to tip the portion of the girder to the left, so that its axis will be at right angles to $b^1\,d^1$, the new position of B D. Now the inclination of $b^1\,d^1$ to B D $= \frac{2\,\overline{\text{B}\,b}}{\text{D}}$. Therefore a point in the section $y\,y^1$ to the left of B D of the girder (Fig. 99) distant x_1 from the centre of A B, will be deflected, due to the bending in this bay, a distance $x_1 \times \frac{2\,\overline{\text{B}\,b}}{\text{D}}$ from the line of A B produced, because the new position of the tangent at B is perpendicular to $b^1\,d^1$ and intersects A B at its middle point.

Now $\overline{\text{B}\,b} = \frac{\text{S}_1}{\text{A}_1 \times \text{E}} \times b$, where S_1 is the stress in A B and A_1 its area, E being the modulus of elasticity, consequently:—

The deflection at the section $y\,y^1$, distant x_1 from the centre of A B, due to the bending of A B $= \frac{2\,b}{\text{E} \times \text{D}} \times \frac{\text{S}_1}{\text{A}_1} \times x_1$.

Similarly the deflection at $y\,y_1$ due to the bending of the next panel $= \frac{2\,b}{\text{E} \times \text{D}} \times \frac{\text{S}_2}{\text{A}_2} \times x_2$, where S_2 and A_2 are the stress in and cross-sectional area of the flange in that bay, and x_2 is the distance from its centre to the section $y\,y^1$—*i.e.* $x_2 = x_1 - b$. It will be seen, therefore, that the total deflection of the centre line at the section $y\,y^1$ relatively to the tangent at $z = \frac{2\,b}{\text{E} \times \text{D}}\ \Sigma\ \frac{\text{S}}{\text{A}}\ x$, the summation extending over the bay lengths between the two sections considered.

Now $\Sigma\ \frac{\text{S}}{\text{A}} \times x =$ the sum of the intensities of stress in the

lengths into which the flange has been divided between the sections at z and y, multiplied by the distance from y of the centre of gravity of the diagram in which the abscissæ are the bay lengths and the ordinates the intensity of flange stress in them.

$$\therefore \Sigma \frac{S}{A} \times x \text{ may be written } = \bar{f} \times N \times k \times X,$$

where $\bar{f}$ is the average value of $\frac{S}{A}$—*i.e.* the average value of the intensity of stress in the lengths of flange between the sections at z and y—X is the distance apart of the sections at y and z, N is the number of bays in that distance, and $k\,X$ is the distance from y of the centre of gravity of the intensity-of-flange-stress diagram.

$\therefore$ the total deflection at y relatively to the tangent at z

$$= \frac{2\,b}{E \times D} \times \bar{f} \times N \times k \times X, \text{ since } N \times b = X, = \frac{2\,k\,X^2 \times \bar{f}}{E \times D}.$$

Similarly the total deflection of z, the centre line at $A\,C$, relatively to the tangent at y, by similar reasoning would be—

$$= 2\,(1 - k)\,\frac{X^2 \times \bar{f}}{E \times D},$$

because the distance of the centre of gravity of the stress diagram from $z = X - k\,X = (1 - k)\,X$.

If $z\,z^1$ (Fig. 99) represents the deflection at z relatively to the tangent at y, and $y\,y^1$ the deflection at y relatively to the tangent at z, since $\frac{y\,y_1}{z\,z_1} = \frac{k}{1 - k}$, $\frac{y_1\,g}{g\,z} = \frac{k}{1 - k}$, in other words, g lies on the vertical through the centre of gravity of the stress diagram. If, therefore, the deflection at one point be known with respect

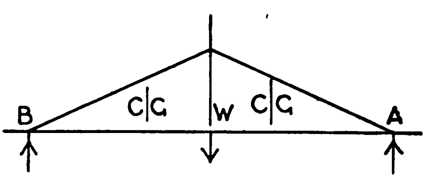

FIG. 100.

to the tangent at another point, the deflection at the second point is found relatively to the tangent at the first by joining the end of the ordinate representing the first deflection with the intersection of the line through the centre of gravity of the

stress diagram and the tangent at the second point and producing the line. Thus, when the deflection $y\,y^1$ is known, by joining y to g and producing to z^1 the deflection $z\,z^1$ is found.

The Inclination of the Tangent at one Point Relatively to that at another Point

The inclination of the tangent at y to that at z

$$= \frac{\overline{y\,y^1}}{k\,\text{X}} = \frac{2\,k\,\text{X}^2 \times \bar{f}}{\text{E} \times \text{D} \times k\,\text{X}} = \frac{2\,\text{X} \times \bar{f}}{\text{E} \times \text{D}}.$$

Take first the case of a girder length L, supported at both ends, with flange of constant area and carrying a load W at the centre. As the flange is of constant area, the intensity-of-flange-stress diagram will be the same shape as the bending-

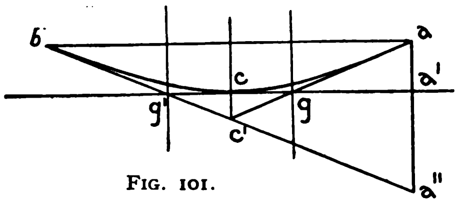

FIG. 101.

moment diagram—*i.e.* a triangle with its apex at the centre of the girder (Fig. 100). For the deflection of the ends relatively to the tangent at the centre (see Fig. 101), if f_c is the stress intensity in the flanges at the centre,

$$\bar{f} = \frac{f_c}{2},\ \text{X} = \frac{\text{L}}{2},\ \text{and } k = \tfrac{2}{3};$$

$$\therefore\ a\,a^1 = \tfrac{2}{3} \times \frac{\text{L}^2 \times f_c}{4\,\text{E} \times \text{D}} = \frac{\text{L}^2 \times f_c}{6\,\text{E} \times \text{D}},\ \text{where } f_c = \frac{\text{W} \times \text{L}}{4\,\text{A} \times \text{D}}.$$

For the deflection at the centre relatively to the tangent at the end, $\bar{f} = \frac{f_c}{2}$, $\text{X} = \frac{\text{L}}{2}$ and $k = \tfrac{1}{3}$, $\therefore\ c\,c^1 = \frac{\text{L}^2 \times f_c}{12\,\text{E} \times \text{D}}$.

For the deflection of one end relatively to the tangent at the other end,

$$\bar{f} = \frac{f_c}{2},\ \text{X} = \text{L and } k = \tfrac{1}{3},\ \therefore\ a\,a'' = \frac{\text{L}^2 \times f_c}{2\,\text{E} \times \text{D}}.$$

The inclination of the tangent at a to that at c

$$= \frac{2\,\text{X} \times \bar{f}}{\text{E} \times \text{D}} = \frac{\text{L} \times f_c}{2\,\text{E}}$$

Next consider the case of a girder length L, supported at both ends, with flanges of constant area and carrying a uniformly distributed load of intensity w. In this case the bending-moment

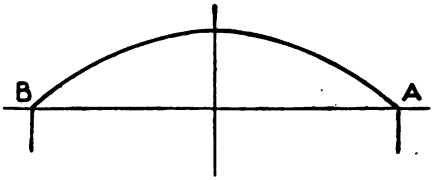

FIG. 102.

diagram and therefore that of the intensity of stress is a parabola (Fig. 102).

For the deflection of the ends relatively to the tangent at the centre (see Fig. 103), if f_c is the stress intensity at the centre,

$$\bar{f} = \tfrac{2}{3} f_c,\ x = \frac{L}{2},\ \text{and } k = \tfrac{5}{8};$$

$$\therefore\ a\,a^1 = \tfrac{5}{8} \times \frac{2\,L^2}{4\,E \times D} \times \tfrac{2}{3} f_c = \tfrac{5}{24} \times \frac{L^2 \times f_c}{E \times D},$$

where $$f_c = \frac{w\,L^2}{8\,A \times D}.$$

For the deflection at the centre relatively to the tangent at the end, $\bar{f} = \frac{2}{3} f_c$, $x = \frac{L}{2}$, and $k = \frac{3}{8}$; $\therefore\ c\,c^1 = \frac{3}{24} \times \frac{L^2 \times f_c}{E \times D}$.

The inclination of the tangent at the ends relatively to that at the centre $$= \tfrac{2}{3}\,\frac{L \times f_c}{E \times D}.$$

Next consider a shore span and cantilever (Fig. 104) with constant flange section, covered with a uniform load of intensity

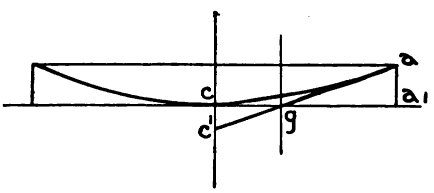

FIG. 103.

w and having a load W at the end of the cantilever arm, w being great enough to give a positive bending moment for some distance from A.

The bending-moment diagram and therefore that of the intensity of flange stress may be constructed by first drawing the skeleton diagram A B^1 D of equivalent concentrated loads at these points—*i.e.* due to a load at D $= \text{W} + \frac{w\,l_1}{2}$, a load at B $= w\,\frac{l_1 + l_3}{2} + w\,l_2$, and a load at A $= \frac{w\,l_3}{2}$, where l_3 is at present not known. Then on A B^1 and B^1 D plot vertically the ordinates of the parabolas, which are the bending-moment diagrams

FIG. 104.

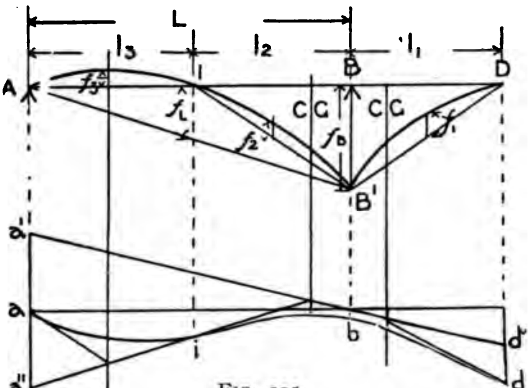

FIG. 105.

for a uniform load intensity w on spans L and l_1 respectively. The point I, where the bending moment changes through zero from positive to negative values, will be a point of inflexion on the deflection curve, as positive bending moments have a sagging effect and negative bending moments a hogging effect. I may be found by taking moments about B. Half the load $w\,l_3$ is transmitted to A, and the other half comes on the cantilever I B D;

$$\therefore\ \frac{w\,l_3}{2} \times l_2 + \frac{w\,l_2^2}{2} = \text{W}\,l_1 + \frac{w\,l_1^2}{2},$$

substituting $\text{L} - l_2$ for l_3,

$$\frac{w}{2}(\text{L} - l_2)\,l_2 + \frac{w\,l_2^2}{2} = \text{W}\,l_1 + \frac{w\,l_1^2}{2},$$

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$$\frac{w\,\text{L}\,l_2}{2} = \text{W}\,l_1 + \frac{w\,l_1^2}{2};$$

$$\therefore\ l_2 = \frac{l_1\,(2\,\text{W} + w\,l_1)}{w\,\text{L}}.$$

In words, l_2 is of such a length that the total load on A B concentrated at the centre of l_2 balances about B the actual load on the right of B.

The deflection at A relatively to the tangent at B is due to the negative triangle of stress A B B^1 and the positive parabola on A B^1.

Due to the negative triangle $\bar{f} = \frac{f_B}{2}$, $X = L$, and $k = \frac{2}{3}$,

$\therefore$ the downward deflection $= \frac{2}{3} \frac{L^2 \times f_B}{E \times D}$.

Due to the positive parabola $\bar{f} = \frac{2}{3} f_L$, $X = L$, and $k = \frac{1}{2}$,

$\therefore$ the upward deflection $= \frac{2}{3} \frac{L^2 \times f_L}{E \times D}$.

The resultant downward deflection of A relatively to the tangent at B therefore $= \overline{a^1 a} = \frac{2}{3} \frac{L^2}{E \times D} (f_B - f_L)$—

Therefore if in Fig. 105 $a\,b$ be drawn $= L$ and $a^1 a$ be set off at a, $a^1 b$ is the tangent at B. If the bending-moment diagram be considered to be constructed by first drawing the skeleton diagram for equivalent concentrated loads at A I B D consisting of the straight lines A I, I B^1, and B^1 D, and then plotting the ordinates of the parabolas for uniform loads on spans l_3, l_2 and l_1, above the lines A I, I B^1, and B^1 D respectively, it is clear that the upward deflection of A relatively to the tangent at I is due to the parabola on A I, then $\bar{f} = \frac{2}{3} f_3$, $X = l_3$, and $k = \frac{1}{2}$,

$\therefore$ the upward deflection $= \overline{a'' a} = \frac{2}{3} \frac{l_3^2 \times f_3}{E \times D}$.

Before the tangent at I can be drawn another point on it is required, and this may be found from the fact that the tangent at I intersects that at B on the line of the centre of gravity of the stress diagram between I and B—*i.e.* of the figure I B^1 B with the curved side I B^1, which is the difference between the triangle I B^1 B and the parabola on I B^1.

The distance of the centre of gravity of this figure from I

$$= \frac{\frac{f_B \times l_2}{2} \times \frac{2}{3} l_2 - \frac{2}{3} f_2 \times l_2 \times \frac{l_2}{2}}{f_B \times \frac{l_2}{2} - \frac{2}{3} f_2 \times l_2}$$

$$= 2\,l_2 \times \frac{f_B - f_2}{3 f_B - 4 f_2}.$$

The tangent at I can now be drawn, and is $a'' i$ in Fig. 105; this enables the tangent at A to be drawn, because it intersects that at I on the line of the centre of gravity of the parabola on A I, which is half-way between A and I. It now only remains to draw the tangent at D, which intersects that at B on the line of the centre of gravity of the figure B D B^1 with the curved side B^1 D. It follows from the last formula that the distance of this point from D $= 2\,l_1 \times \dfrac{f_B - f_1}{3\,f_B - 4\,f_1}$.

The deflection of D downwards relatively to the tangent at B is due to the effect of the negative triangle of stress B^1 D B less the effect of the positive parabola on B^1 D.

Due to the negative triangle $\bar{f} = \dfrac{f_B}{2}$, $x = l_1$, and $k = \frac{2}{3}$,

$\therefore$ the downward deflection $= \frac{2}{3}\,\dfrac{l_1^2 \times f_B}{E \times D}$.

Due to the positive parabola $\bar{f} = \frac{2}{3} f_1$, $x = l_1$, and $k = \frac{1}{2}$,

$\therefore$ the upward deflection $= \frac{2}{3}\,\dfrac{l_1^2 \times f_1}{E \times D}$.

The resultant downward deflection of D relatively to the tangent at B therefore $= d^1 d = \frac{2}{3}\,\dfrac{l_1^2}{E \times D}\,(f_B - f_1)$.

The form of the curve of deflection can therefore be completely traced.

If the intensity of stress curve is not composed of straight lines or parabolas, the average value of the stresses and the distance of the centres of gravity of the stress diagrams are not so readily found, but the method is still applicable.

Prof. T. Claxton Fidler, in a paper* to the Institution of Civil Engineers, showed how use may be made of the deflection curve to find the bending moments in continuous girders, and has further developed the method in his valuable treatise on "Bridge Construction," † where he gives a very elegant geometrical solution of the problem.

The deflection at any point of a girder with continuous web under any specified load may be found by imagining a load P placed at the point where the deflection is required, and then

* *Proceedings Inst. C.E.*, vol. lxxiv.

† "A Practical Treatise on Bridge Construction," T. Claxton Fidler. Charles Griffin & Co.

supposing the actual load on the bridge to be gradually applied. The load P will then do work by moving through a distance u, the deflection at the point under the actual load. The actual load gradually applied to the bridge would do work $= \Sigma \frac{W \delta}{2}$ where W represents any part of the actual loads on the bridge and δ the distance through which the centre of gravity of W deflects when the actual load is gradually applied.

The total external work will therefore $= P u + \Sigma \frac{W \delta}{2}$.

If M^1 be the bending moment at any section due to the load P, and p the intensity of stress, in the section, on a horizontal strip at a point distant y from the neutral axis, where the breadth $= z$, the stress on this strip due to $P = p\, z\, d\, y$.

If q be the intensity of stress caused in the same strip by the actual load by itself, the fibres of the strip of length $d\,x$ will be extended by the application of the actual load by an amount $= \frac{q}{E}\, d\,x$. The increase of stress on the strip due to the actual load $= q\, z\, d\, y$, thus the total stress on the strip will then be $(p + q)\, z\, d\, y$.

The work done by the stresses on the strip when the actual load is gradually applied = the mean stress $\times$ the extension due to the actual load,

$$= \frac{\overline{p + q + p}}{2} \times z\, d\, y \times \frac{q}{E}\, d\, x = \left(p + \frac{q}{2}\right) \frac{q\, z}{E}\, d\, x\, d\, y.$$

If p_1 be the intensity of stress at the outside fibre, distant d from the neutral axis, due to the load P, and q_1 be the intensity of stress there, due to the actual load, then

$$p = p_1 \frac{y}{d} \text{ and } q = q_1 \frac{y}{d};$$

$\therefore$ the work done by the stress on the fibre

$$= \left(p_1 + \frac{q_1}{2}\right) q_1 \frac{y^2 z}{d^2 \times E}\, d\, x\, d\, y.$$

Integrating this expression over the depth of the beam and over its length :—

The work done by the stresses in the girder

$$= \int_0^L \frac{\left(p_1 + \frac{q_1}{2}\right) q_1}{d^2\, E}\, d\, x \int_{d^1}^{d} y^2\, z\, d\, y.$$

Now $\int_{d^1}^{d} y^2 z\, d\, y = \text{I}$, the moment of inertia of the cross-section about the neutral axis, and if M is the bending moment due to the actual loads, we have

$$\frac{p_1}{d} = \frac{\text{M}^1}{\text{I}} \text{ and } \frac{q_1}{d} = \frac{\text{M}}{\text{I}},$$

$\therefore$ the formula may be written—

$$\int_0^{\text{L}} \frac{\left(\text{M}^1 + \frac{\text{M}}{2}\right) \text{M} \times \text{I}\, d\, x}{\text{I}^2\, \text{E}},$$

or the total work done by the stresses in the girder

$$= \int_0^{\text{L}} \frac{\text{M}^1\,\text{M}}{\text{E}\,\text{I}}\, d\, x + \int_0^{\text{L}} \frac{\text{M}^2}{2\,\text{E}\,\text{I}}.$$

Now $\Sigma\ \frac{\text{W}\,\delta}{2} = \int \frac{\text{M}^2}{2\,\text{E}\,\text{I}}$, the resilience of the beam due to the actual load;

$$\therefore \qquad \text{P}\,u = \int_0^{\text{L}} \frac{\text{M}^1\,\text{M}}{\text{E}\,\text{I}}\, d\, x,$$

or

$$u = \frac{1}{\text{P}} \int_0^{\text{L}} \frac{\text{M}^1\,\text{M}}{\text{E}\,\text{I}}\, d\, x.$$

P may, of course, be taken equal to unit load, say 1 ton. It should be noted that P need not be a vertical force, but, in the case of such a structure as an arch, for instance, in which other than vertical forces act, may be taken as a horizontal force applied at the hinge.

Deflection of Braced Girders

In the case of braced girders, as already pointed out, the members of the bracing are either in tension or compression, and must therefore lengthen or shorten as the case may be; this fact affects the new shape taken up by the panel, and there will be a greater tendency for part of the bending to take place at the joints.

Taking the same lett· 106 as for the girder

with continuous web and greatly exaggerating the strains as before, it will be seen that if the diagonal meeting the flange in compression extends by an amount B c and the vertical C A shortens by an amount A A^1, and a circle centre C and radius C c be described to intersect the circle centre A^1 and radius A^1 b at B^1, then B^1 will be the new position of the panel point B instead of b^1 as in the previous case, the deflection being therefore greater by an amount b^1 B^1 than in the case of the continuous web. By setting off B^1 D^1 equal to the shortened length of B D, the new position of D is obtained. As the deformations are very slight compared with the dimensions of the structure, the circle centre C and radius C c will coincide with the line c B^1, perpendicular to C c, and the circle centre A^1 and radius A^1 b will coincide with the vertical line B^1 b. If, instead of using diagonals normally in tension as in the Pratt girder, the alternative diagonals are introduced which would be in compression, it will be seen from Fig. 106 that the deflection for the panel would not be so great, and it would be nearly the same as for a girder with continuous web. For, considering the panel point D, the diagonal A D will shorten with the vertical, but cosec θ times the amount—*i.e.* D will be brought to $d_1{}^1$, hence its displacement D $d_1{}^1$ will be less than that of B for a tension brace, viz. B B^1. If instead of the brace C B being in tension, the counterbrace is brought into play, the displacement due to the diagonal must be plotted downwards from b^1 to b'', instead of upwards to B^1. This would reduce the displacement of B for the panel, and not increase it. It will be noticed that the displaced position of B D in this case—viz. b'' d''—will be parallel to B^1 D^1, so that although the deflection for the individual bay is reduced, the effect of the deformation of the bay on the deflection of those to the left will not be altered. Therefore, in the

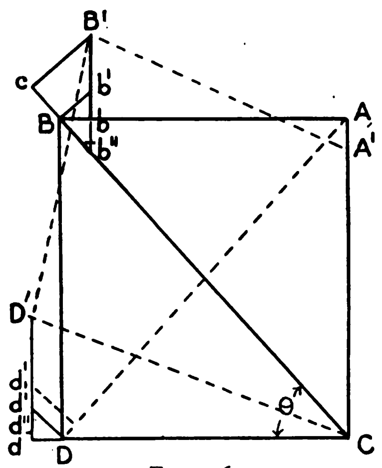

FIG. 106.

bays in which the counterbraces are brought into play, the deflection will be reduced for those bays only, and the amount shortly to be determined must be deducted from the total deflection.

Take the case in which the tension braces are brought into action, to prove first that although the deflection at the bay is increased by the amounts $B^1 b^1$, yet the inclination of $B^1 D^1$, which rules the effect of the bay in producing deflection in the bays to the left, is the same as before. The vertical $B^1 D^1$ is shorter than $b^1 d^1$ by the amount the vertical contracts, which is $\frac{S^1}{A^1} \times \frac{D}{E}$, S^1 and A^1 being the stress in and the cross-sectional area of the vertical, but its ends still lie on the vertical lines $B^1 b$ and $D^1 d$. The altered inclination therefore is

$$\frac{2\,\overline{B\,b}}{D\left(1 - \frac{S^1}{A^1 E}\right)}, \text{ and } B\,b = \frac{S^1}{A^1} \times \frac{b}{E};$$

∴ the altered inclination

$$= \frac{2\,\frac{S^1}{A^1} \times \frac{b}{E}}{\frac{D}{E}\left(E - \frac{S^1}{A^1}\right)}.$$

Now $\frac{S^1}{A^1}$ is negligible in comparison with E, consequently the inclination is $2\,\frac{S^1}{A^1} \times \frac{b}{E \times D}$ as in the case of the continuous web. The deflection will therefore be the same as for the continuous web plus the amount $B^1 b^1$ for each bay in which the brace is in tension.

Now $B^1 b^1 = \frac{B\,c}{\sin\theta} = \frac{D \times S^1}{\sin^2\theta \times E \times A^1} = \frac{B \times S^1}{E \times A^1}$, where $B = \frac{D}{\sin^2\theta}$, S^1 is the stress in the diagonal and A^1 its cross-sectional area. The effect of the deformation of the bracing is therefore to increase the deflection by $\frac{B \times S^1}{E \times A^1}$ for the one bay or by $\frac{B}{E}\,\Sigma\,\frac{S^1}{A^1}$, for the number of bays $= N$ between the two sections considered.

The total deflection of the one section relatively to the other therefore

$$= \frac{2\,b}{E \times D}\,\Sigma\,\frac{S}{A}\,x + \frac{B}{E}\,\Sigma\,\frac{S^1}{A^1}.$$

If $\frac{S}{A} = f$, $\bar{f}$ is the mean intensity of stress in the flange, and

if $\frac{S^1}{A^1} = f^1$, $\overline{f^1}$ is the mean intensity of stress in the diagonals in the N bays, whose total length is X, therefore :—

$$\text{The total deflection} = \frac{2\,b}{E \times D} \times N \times \overline{f} \times k\,X + \frac{B}{E} N \overline{f^1}$$

$$= \frac{2\,k\,X^2 \times \overline{f}}{E \times D} + \frac{B}{E} N \overline{f^1},$$

$k\,X$, as before, equals the distance from the point where the deflection is required to the centre of gravity of the diagram, whose abscissa represents N bay lengths and the ordinates represent the flange stresses in those bays. Thus N times $\frac{B}{E} \times \overline{f^1}$ must be added to the deflection found in the case of the continuous web girder, and this quantity is the extra deflection due to the web members. But when the counterbraces in any bays are strained instead of the braces, the second term must be subtracted for those bays instead of added.

The deflections may be obtained graphically; in order to allow of them being drawn on a much larger scale than the

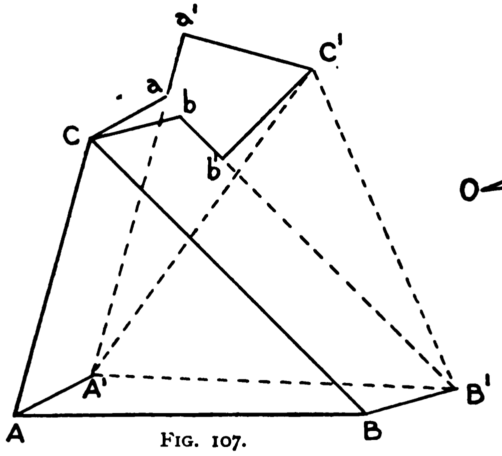

FIG. 107.

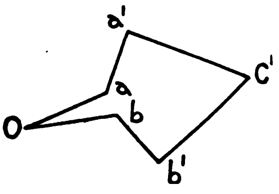

FIG. 107*a*.

elevation, it is more convenient to draw a diagram of displacements only. This can easily be done, for suppose in Fig. 107 A B C represents three panel points in a bridge, and let A A^1, B B^1 be the displacements of A and B, and let A C be lengthened by an amount δ and let B C be shortened by a length δ^1.

To find the new position of C, it is only necessary with centre ^{1} to draw the arc of a circle radius (A C + δ), and with centre ^{1} describe the arc of a circle radius (B C − δ^1), the intersection ill of course be C^1, the new position of C. By reference to Fig. 107 t will be seen that this is equivalent to drawing C a parallel and qual to A A^1, C b parallel and equal to B B^1, producing A^1 a the ength δ to a^1, and shortening B^1 b the length δ^1 to b^1, and from a^1 and b^1 drawing lines perpendicular to $a\,a^1$ and $b\,b^1$ (which is equivalent to describing the arcs of the circles, centres A^1 and B^1, since the displacements are small). These perpendiculars intersect at C^1. This construction can be effected away from the actual elevation; for take any pole O (Fig. 107a), from O draw lines parallel and proportional on a suitable scale to A A^1 and B B^1, the displacements of A and B, and from the ends of hese lines draw lines equal to and in the same directions as the lterations in the lengths of the two members meeting at C, nd from the extremities of the latter draw perpendiculars which intersect at C^1, then it is obvious that O C^1 is the displacement of C in direction and magnitude on the scale employed.

Applying this to the bay we ave been considering, let C e the pole which represents successively the initial positions of the oints B and D whose displacements re to be found. In Fig. 106, C is supposed to remain fixed, A is displaced long A C to A^1; first find the displacement of B. Following the directions bove given from C (Fig. 108), draw ines parallel and equal on a convenient cale to the displacements of C and , as C does not move C represents ^{1} and C A^1 represents the displacement A A^1 (Fig. 106). From C^1 and A^1 raw lines C^1 c and A^1 b parallel and qual to B c and B b (Fig. 106), the lterations in length of C B and A B espectively, and at the extremities of hese lines erect perpendiculars which ntersect at B^1. C B^1 therefore is the displacement of B in agnitude and direction.

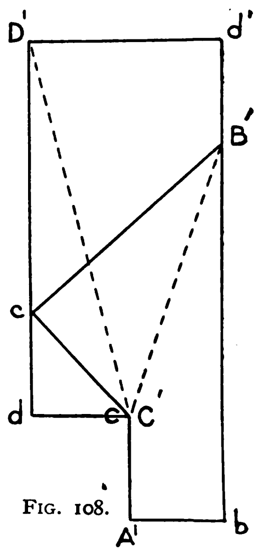

FIG. 108.

Next, to find the displacement of D knowing those of C and B.

From C draw lines parallel and equal to the displacements of C and B; that of C is nothing, as before, therefore C^1 is at C, that of B is C B^1 already drawn. From the ends of such lines draw lines C d and $B^1 d^1$ parallel and equal to the alterations in length of C D and B D respectively, and draw perpendiculars, which intersect in D^1, at the extremities of these lines; therefore C D^1 is the displacement of D in magnitude and direction. The same process may be continued for any number of panels.

The deflections at a section of a braced girder due to any specified load can be found in a similar manner to that explained for girders with continuous webs. If a load P be applied at the point where the deflection is required, and this load produces a stress s in any member of the girder, and if then the actual load be considered to be gradually applied to the structure which is capable by itself of producing a stress S in any member, P will deflect a distance u due to the actual load, and each member will lengthen or shorten, as the case may be, a length $\frac{S}{A \times E} \times L$, where A is the cross-sectional area of the member under consideration and L is its length. Now the stress in the member before the actual load is applied was s, and after its application it is $S + s$, therefore the internal work done in the member when the actual load is gradually applied

$$= \frac{S + 2s}{2} \times \frac{S \times L}{A \times E} = \frac{S^2 \times L}{2A \times E} + \frac{S \times s \times L}{A \times E}.$$

The work done by the external loads

$$= P \times u + \Sigma \frac{W \times \delta}{2},$$

where W represents one portion of the actual load and δ the distance through which the centre of gravity of that portion of the load is deflected, and Σ represents the summation for each portion of the load.

Since the external work done on the girder equals the work done by the stresses in the members—

$$P \times u + \Sigma \frac{W \times \delta}{2} = \Sigma \frac{S^2 \times L}{2A \times E} + \Sigma \frac{S \times s \times L}{A \times E}.$$

Now $\Sigma \frac{W \times \delta}{2} = \Sigma \frac{S^2 \times L}{2A \times E}$ from the formula for resilience.

$$\therefore \quad P \times u = \Sigma \frac{S \times s \times L}{A \times E}$$

$$\text{or} \quad u = \frac{1}{P} \Sigma \frac{S \times s \times L}{A \times E}.$$

P may of course be taken as unit load, say 1 ton.

Before this formula can be applied, it is of course necessary that the cross-sectional area of the members, the stress S in each member due to the actual load, and the stress s due to the load P, should be known.

It will be observed that u is the deflection in the direction of the force P applied at the point where the deflection is required. If, in the case of an arch, hinged at one point of support and capable of sliding horizontally at the other support, the actual load consists of a single load Q, and unit force be applied at the free support, the horizontal displacement at that support $= \Sigma \frac{S \times s \times L}{A \times E}$, S being the stress in any member length L, of cross-sectional area A, due to the actual load Q, and s the stress in the same member due to unit horizontal force at the free support.

If the actual load be changed from Q to unity at the same point of application, the stress in the member would be $\frac{S}{Q}$, and if the horizontal force applied at the free hinge be increased to Q the stress in the member due to it would be Q s; the horizontal displacement at the free support then equals

$$\frac{1}{Q} \Sigma \frac{S \times Q \times s \times L}{Q \times A \times E} = \frac{1}{Q} \Sigma \frac{S \times s \times L}{A \times E}.$$

If the actual load is a horizontal load Q at the free hinge, the stress due to the actual load would be Q s, and therefore the horizontal displacement of the hinge would be $Q \Sigma \frac{s^2 \times L}{A \times E}$.

And it may be noticed that if the actual load be Q at the free hinge and the deflection is required at the point of the girder where a unit load is applied, the stress in a member due to Q at the hinge = Q s, and the stress due to unit load on the girder $= \frac{S}{Q}$, therefore the deflection at the point of application of the load unity due to the horizontal force Q at the free hinge $= \Sigma \frac{S \times s \times L}{A \times E}$.

Camber of Girders

In order to give a girder such a camber that when the live load is in the position that produces the maximum moment at the centre, the straight flange shall be truly horizontal, it is only necessary to make the tension and compression members in the flange and web shorter or longer respectively by the amount $\frac{S \times L}{A \times E}$, s being the stress in the member considered due to the dead load and to the above disposition of the live load. When the live load comes into the position indicated, the total stresses in the members will then cause their lengths to be such that there will be no deflection under the load.

CHAPTER X

ARCHES

The arch is considered before the suspension bridge to avoid an artificial difficulty which often interferes with the proper conception of the stresses in an arch if regarded as an inverted suspension bridge, owing to the fact that in the latter a flexible parabolic member can be used to take the tensile stress whilst the resultant bending moment is taken up independently by a stiffening girder.

METAL ARCH

The difference between the problem of design of an arch hinged at the two supports and that of a curved girder supported at the two ends is that in the arch there is a horizontal component of the reactions which acts in the line of the hinges; thus the forces which maintain the positions of the ends of the curved member are applied externally to the structure, instead of by the straight member connecting the extremities of the curved member in a bowstring girder. Consequently the bending moment at any section equals the moment about its axis of the vertical component of the reaction on the right, of any loads between the section and that reaction and of the horizontal component of that reaction. The moment of the two latter forces will of course be of the opposite sign to that of the vertical component of the reaction, because they tend to rotate the portion of the arch to the right of the section in the opposite sense to what the vertical reaction does.

It is obvious that the arch, corresponding as it does to the curved member of a bowstring girder in its upright position, is primarily in compression due to the loads.

It is necessary in this case, as it is for a girder, to determine the stresses due to dead load and those due to live load, and in the latter case it has to be first ascertained what positions of the moving loads will produce the maximum stresses in each member.

When the dead load reactions are known the equilibrium polygon can be drawn for the dead load, and from that the dead load stresses in all the members can be determined. For the moving load, it is best to consider the effect of a single load at the different panel points in order to be able to settle which panel points should be loaded to produce the maximum tension and compression in each member. In both cases it is necessary to determine the vertical and horizontal components of the reactions. For arches hinged at the supports the vertical reactions may be obtained by taking moments about either hinge. Since the horizontal components of the two reactions are the only horizontal forces acting on the structure, these must be equal and opposite, and therefore their line of action will pass through the two supports. Consequently the vertical components of the reactions will be inversely proportional to the distance of the resultant of the load from the two supports, as in the case of a girder. In the case of an arch without hinges this is not so, because there is then a bending moment at each support in addition to a reaction, just as in the case of the intermediate piers of a continuous girder, and, as in that case, although the sum of the vertical components of the reactions is equal to the load on the span, their relative values are altered from what they would be if the arch were supported and not fixed at the springings.

The finding of the horizontal component of the reactions in the case of the three-hinged arch presents no difficulty, because the introduction of the central hinge renders it a statically determinate structure for both vertical and horizontal forces. In the case of arches hinged only at the points of support, the horizontal component of the reactions is not so readily found, because the structure is not statically determinate for horizontal forces, but the theory of elasticity has to be invoked for the purpose. It is usual in this case to neglect the displacements due to shear, and in the first instance also those due to axial thrust, as these are comparatively insignificant compared with the displacements due to bending. For, as already pointed out in the case of girders, the effect of the bending of each element of length is magnified at points distant from it in proportion to their distance, whereas the strains due to shear and axial thrust are not magnified. The horizontal reaction is determined by expressing, in the form of an equation, the fact that although the bending of each element of length tends to move one point

of support, if the other is considered immovable, relatively to the latter, since the points of support are really a fixed distance apart, the sum of the displacements of the point of support due to the bending of all the elements of length into which the arch is divided must be zero.

In the case of an arch without hinges two more equations are necessary to find the unknown moments at the abutments, and these are forthcoming, because just as the horizontal displacements of one point of support mutually cancel each other, so also do the vertical placements ; also, since the ends are fixed, the inclination of the tangent to the centre line of the arch at each end is constant, notwithstanding the load. This last fact can be expressed as an equation because it necessitates the total change of inclination of the centre line integrated from end to end of the arch being zero. The difference when the supports are fixed, which involves there being a couple there in addition to the reaction, is seen in Fig. 119 ; there R^1 is the reaction at the right support, and if the arm of the couple is altered so that the two forces constituting it are both equal to R^1, and if, further, the couple is placed so that the force of the couple equal and opposite to the reaction is in line with it, it is then evident that these two forces counteract each other, and a force equal to R^1 but at a different level is left.

Thus the moment of the reaction at one support about the other equals the moment of the load minus the moment of the horizontal component of that reaction, because the horizontal components of the two reactions are no longer in the same line.

If V be the vertical component of the reaction at either support if both were hinged, and V^1 is its value when both supports are fixed, $(V - V^1) \times L = H \times$ the vertical distance between the horizontal components of the two final reactions, L being the span and H the horizontal reaction.

THREE-HINGED ARCH

Dead Load Stresses.—When an approximate value of the dead load is known, the value of each panel load can be determined and the equilibrium polygon can be drawn. The load being symmetrical, the reactions would be the same if one-half the load be taken at the centre and one-quarter at each support. Drawing first the force diagram (Fig. 109), calling W the total dead load, set off the panel loads on the line $z^1 z$, starting at the top with the load at B, so as to take the actual

in order in the same sense—clockwise in this case; then mark off $z^1 c^1 = \frac{W}{4}$, $c^1 c = \frac{W}{2}$, and $c z = \frac{W}{4}$, which corresponds to taking half the load on each single rib at each end of it, which is an equivalent load to the actual panel loads as far as the reactions at A, C, and B are concerned. From c and c^1 draw lines parallel to A C and B C, then O is the pole of the force diagram; O c^1 represents the reaction of the rib B C on A C, similarly c O that of the rib A C on B C; z O is the resultant reaction at A, and O z^1 is the resultant reaction at B, and the horizontal line O H is the horizontal component H of the reactions, and of the resultant force at each section of the arch. Starting from B, draw lines parallel to O z^1 and O b till the latter

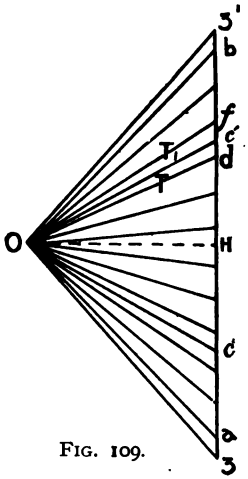

FIG. 109.

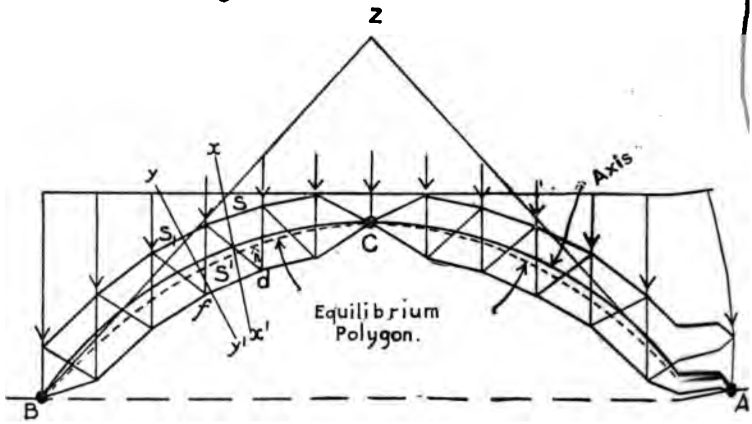

FIG. 110.

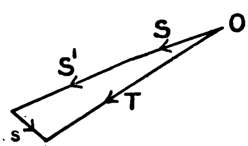

FIG. 111.

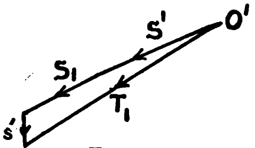

FIG. 112.

meets the line of action of the load at the next panel point, and from this intersection draw a line parallel to

the next ray, and so on. Similarly from A; the equilibrium polygon will of course pass through C. To find the dead load stresses in the upper chords, take a section line intersecting the chord in question and two other members—*e.g.* the dotted section $x\,x^1$; the other two members intersected meet at d, which is therefore the centre of moments for the chord member under consideration—*i.e.* by taking moments about this point the stresses in the other two members are eliminated. The stress S in the chord multiplied by p, the perpendicular from d on it equals the force T (represented by the ray O d in the force diagram) multiplied by r, the perpendicular from d on the corresponding line in the equilibrium polygon; *i.e.* $S \times p = T \times r$. It will be observed that T is the resultant force acting across the section taken, therefore $T \times r$ is the moment of all the external forces acting on the portion of the rib between the section and B; and T is of course acting downwards to the left on this severed portion of the rib. The stress in the chord will therefore also be acting downwards to the left, and it is the stress exerted by the portion of the chord to the right of the section on the portion of the chord to the left of the section—*i.e.* it is compressing the section, or the chord is in compression. In a similar manner the dead load stresses can be found for all chord members. With respect to the braces, if the equilibrium polygon coincided with the axis there would be no stress in them except that due to the rib shortening; but if they do not coincide the stress may readily be found when the chord stresses are known. For this purpose let sections $x\,x^1$, $y\,y^1$ be taken, cutting the brace in which the stress is required and two members of the chord; then it must be remembered that we are considering the equilibrium of the severed portion of the rib to the left of the section. At the section $x\,x^1$ the resultant force, represented in position by the line of the equilibrium polygon and in magnitude by the corresponding ray in the force polygon, must be equivalent to the resultant of the stresses S and S^1 in the two chords, and the stress s in the inclined brace cut by the section and acting across the section from right to left, because we are considering the action of the portion to the right on that to the left; therefore by drawing from O (Fig. 111) lines parallel and equal to S and S^1 and to T, the closing line s represents the stress in the severed brace; as drawn it is a tension because the part of the brace he right of the section is pulling away from it. In taking

the section $y\,y^1$ to cut a vertical, it must be observed that the load acting at its upper extremity does not act upon the severed portion of the rib to the left, therefore the section must be drawn to cut the equilibrium polygon in the bay to the left of the vertical. Drawing from O^1 (Fig. 112) lines parallel and equal to S^1 and S_1 and to T_1, the closing line s^1 is the stress in the vertical; as drawn it is a compression, because the portion of the vertical to the right of the section is thrusting downwards on the section. All the dead load stresses may be thus determined.

It is instructive to observe that the equilibrium polygon is the bending-moment diagram for the loads and vertical reactions; its ordinates measured on the linear scale to which the elevation of the arch is drawn $\times$ H in tons gives the moment of the vertical forces in tons-feet. Now if moments be taken about any point on the axis of the arch, the resultant moment = the moment of the vertical forces minus the moment of the horizontal component of the reaction. If y is the ordinate of the axis of the arch at the point in question, this latter term = $H \times y$. That is to say, the ordinates of the axis, measured with the same scale used to measure the ordinates of the equilibrium polygon to obtain the moments of the vertical forces, give the moment of the horizontal component of the reaction. Consequently the difference between the ordinate at any point of the equilibrium polygon and that of the axis at the same point, measured with the moment scale, equals the resultant bending moment at that point of the axis. If the equilibrium polygon coincides with the line of the axis, there would be no bending moment, but simply a compression of equal intensity in the upper and lower chords and a shear, but the effect of non-coincidence involves the existence of a resultant bending moment, which increases the compression in the upper or lower flanges according as this moment is positive or negative. Instead, however, of taking moments about the axis and finding the stress in the chords due to direct compression and then that due to the resultant bending moment, it is more direct to take moments about the centre of moments of the member in question—*i.e.* the point where the other two members cut by the section meet, as explained with respect to the chord member in Fig. 110—when the stress is obtained at one operation.

Another way of looking at the above is the following:—The tangent to the equilibrium polygon at any vertical section is the resultant T of the external forces at that section, its amount

being the length of the corresponding ray in the force polygon; if therefore two equal and opposite forces be applied at the axis of the arch in this section, parallel to this resultant force and equal to it in magnitude, instead of the single force applied at the point where the equilibrium polygon cuts the vertical, we may now substitute an equal and parallel force at the axis and a moment equal to the original force multiplied by the perpendicular q upon it from the axis. This force acting at the axis would have a component along the tangent to the axis, causing thrust, and a component normal to this, causing shear, and the moment $\text{T} \times q$ would evidently be equal to the constant horizontal component H, of the resultant force multiplied by the vertical distance z of its point of intersection with the vertical section from the axis—*i.e.* $\text{T} \times q = \text{H} \times z$, because if θ is the inclination of the tangent to the equilibrium polygon at the section, $\text{H} = \text{T} \cos \theta$, and z is perpendicular to H and q to T; $\therefore q = z \cos \theta$.

Live Load Stresses.—It is necessary to find at which panel points loads must be placed to produce tensions in any particular member, and at which panel points they must be placed to produce compressions in it. Then to find the maximum tension in that member all the former joints must be taken as loaded, and to find the maximum compression in it all the latter joints must be taken as loaded.

If we find the expressions for the vertical and horizontal components of the reactions for a load at any panel point distant x from the nearest support A, the values of these for any combination of panel loads can be obtained by simply adding the results for the individual panel loads. Let $\text{A B} = \text{L}$, and the depth from C to A B equal D. If the load Q on A C be distant x from A, the vertical reaction at A, V_A, due to it, equals $\text{Q}\,\dfrac{\text{L} - x}{\text{L}}$, and the vertical reaction at B_B, V_B, $= \text{Q}\,\dfrac{x}{\text{L}}$, as found by taking moments about B and A respectively. Since there is no load on B C, the reaction at B must pass through C, and therefore its horizontal component $= \text{H} = \dfrac{\text{Q}\,x}{\text{L}} \times \dfrac{\text{L}}{2\,\text{D}} = \dfrac{\text{Q}\,x}{2\,\text{D}}$. To illustrate how to find the panel points at which loads produce tension or compression in a chord member, consider the stress in $d\,f$ (Fig. 113). For this purpose take a section $x\,x^1$, cutting $d\,f$ and two other members; e is then the centre of moments for $d\,f$. Consider the momer' ces acting on the severed

part of the rib to the right of this section. For a load between A and e—*e.g.* at g—we have acting on the portion of the rib considered the weight at g and the reaction due to it at A; the resultant of these two is a force acting along C B, because the

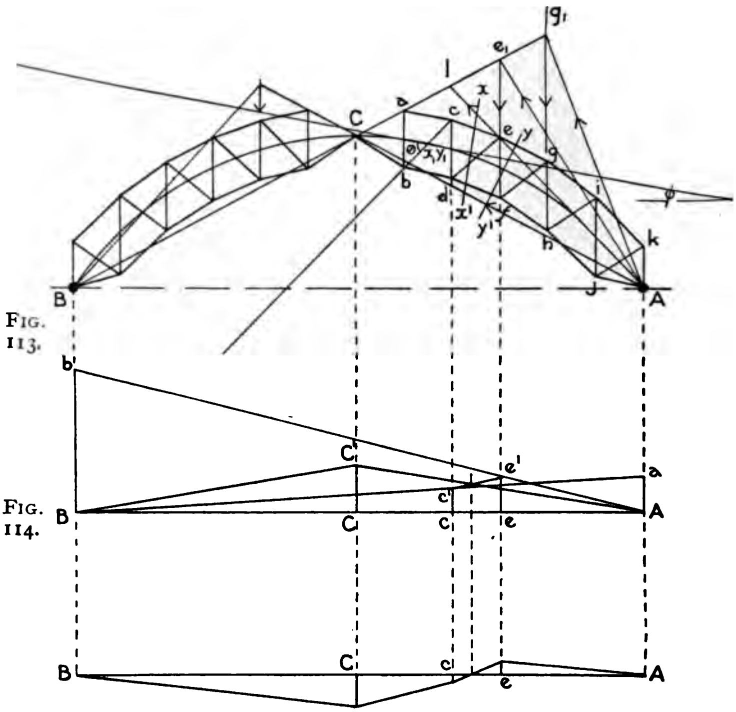

FIG. 113.

FIG. 114.

FIG. 115.

weight and the two reactions balance, and therefore intersect at the same point g_1, which lies on B C produced, because there is by supposition no other force acting on that rib. The moment of this force is positive, as it tends to turn about e in the contra-clockwise sense, thus the stress in $d\,f$ to the right of the section has a positive moment about e; loads from A to e therefore put $d\,f$ in tension. When the load is taken to the left of e it no longer acts on the portion of rib e A, and it only affects it through its reaction at A; that reaction has a positive moment about e until its line of action passes through e; *e.g.* for a load at e the reaction at A is A e_1, the moment about e being positive. When the reaction at A passes through e the load is

at *l*, and the moment about *e* will then of course be zero. When the load is between *l* and C the moment of the reaction will be clockwise or negative, and loads at such panel points will cause a compression in *d f*. For loads on B C the reaction at A will be along A C, producing a negative moment about *e*. Thus loads from A to *l* give a positive moment about *e* and a tension in *d f*, and loads from *l* to B give a negative moment about *e* and a compression in *d f*. Similarly for the other chord members. For the upper chords, moments must be taken about the lower panel points. In case of a panel point lying below A C, as in the case of *b*, the moment of the reaction for loads on B C will be positive, and therefore in this case *a c* will be in compression for loads at all the joints.

To show how to find at which panel points loads should be placed to produce tensions or compressions in a diagonal brace, consider the diagonal *d e* and take a section $x\,x^1$ cutting it and the chords *c e* and *d f*, and consider the forces acting on the severed portion of the rib to the right of this section. The centre of moments for *d e* would be the point where *c e* and *d f* meet, but if these chords are parallel the point will be at an infinite distance, but this will not prevent the sense of the moment of the acting forces being determined. Consider the point in *c e* produced to the left. For loads from A to *e* the resultant force acting on the portion $A\,x\,x^1$ of the rib would act along C B, because it is the resultant of the weight and the reaction at A due to it, and this would give a negative moment; therefore for such loads *d e* would be in compression, as the stress on the section $x\,x^1$ from the right must act to the left; loads from *c* to C only affect the portion of the rib under consideration by the reaction they cause at A, and this would give a positive moment about a point in *c e* produced to the left, therefore the stress in *d e* must be away from the section on the right—in other words, it is in tension. For loads on B C which produce a reaction at A in the direction A C, the moment will remain positive—*i.e.* the stress in *d e* will be tension. Thus *d e* is in compression for loads to the right of *e* and in tension for loads to the left of *c*.

Each diagonal must be considered separately, for in case of the diagonal *h i*, for instance, the centre of moments will be above A C, therefore the moments for loads on B C will be negative.

Next consider the case of a [illegible] Take *c f* as an example,

the centre of moments will be at the intersection of $d\ f$ and $e\ g$, and since $e\ g$ is steeper than $d\ f$, this point will be to the right above A C produced. The moment of the resultant for loads at g, i, k, and their respective reactions is positive, and the stress in $e\ f$ would be a tension, because it must be downwards from the section; for loads from e to B the moment of the reactions would be negative, and the stress in $e\ f$ therefore a compression, because it must be acting upwards towards the section. It will be observed that for the vertical $i\ j$, for instance, the centre of moments would be below A C, therefore the moments of the reaction of loads from B to C would be positive.

Having determined in this manner at which panel points loads must be placed to give maximum tensions and compressions in each member, the stresses in the members for such loads may be found (1) by drawing the stress diagram for each arrangement of load, since the structure is statically determinate; (2) by taking moments about the centre of moments for each member. To illustrate the second; supposing V_A and H be found for each of the loads which causes say a tension in any given member. Take A as origin and A B as axis of x. Let $\xi\ \eta$ be the co-ordinates of the centre of moments for that member. It would be a panel point for the chord members, but not for the web members. x is the distance of any of the loads which come between A and the section severing the member in question, and p is the stress perpendicular of the member—*i.e.* the perpendicular from the centre of moments on to the member.

Then if S is the stress in the member, $S \times p = \xi\, \Sigma V - \eta\, \Sigma H - \Sigma Q\, (\xi - x)$. A difficulty presents itself in connection with web members between parallel chords, as in this case the centre of moments is at an infinite distance; but this is only an apparent trouble, because the ratios $\frac{\xi}{p}$ and $\frac{\eta}{p}$ have definite values in such a case, as is evident from the following, and x becomes negligible in comparison with ξ. If r is the distance of the point ξ, η from the centre of the diagonal $b\ c$ (Fig. 113), whose co-ordinates are $x_1\ y_1$, say, produce the centre line of this bay, and let it be inclined to the axis of x at an angle ϕ.

Then $\xi = r \cos \phi + x_1$, $\eta = r \sin \phi + y_1$ and $p = r \sin \theta$.

$$\therefore \qquad \frac{\xi}{p} = \frac{\cos \phi}{\sin \theta} + \frac{x_1}{r \sin \theta}, \ \frac{\eta}{p} = \frac{\sin \phi}{\sin \theta} + \frac{y_1}{r \sin \theta}.$$

ut since r is infinite, the second terms vanish.

$$\therefore \quad \frac{\xi}{p} = \frac{\cos\phi}{\sin\theta} \text{ and } \frac{\eta}{p} = \frac{\sin\phi}{\sin\theta};$$

$\therefore$ the stress in such a member

$$= \frac{1}{\sin\theta}\{\cos\phi\,(\Sigma\,\mathrm{V} - \Sigma\,\mathrm{Q}) - \sin\phi\,\Sigma\,\mathrm{H}\}.$$

Instead of solving the problem of finding the stress in any member in two steps by first finding at which panel points the loads must be placed, and then taking moments, a diagram may be drawn for each member giving the stress in that member due to a load at each panel point consecutively. Then by adding the ordinates at the various panel points at which a load produces a stress of the same sign in the member, the maximum stress of that sign in the member is determined. Such a diagram is called an "influence" diagram. For this purpose, take as before a section cutting the member in which the stress is required and two other members, so as to sever the rib, and consider the forces acting on the part of the rib between the section and A. Taking A for the origin and A B the axis of x, call ξ, η the co-ordinates of the centre of moments for any member—*i.e.* the point of intersection of the lines of the other two severed members, produced if necessary; p the stress perpendicular—*i.e.* the perpendicular from the centre of moments on to the member in question; M the moment about the centre of moments of the external forces, acting on the portion of the rib severed by the section, due to a single load Q at a panel point distant x from A. The stress in the member due to this load $= \frac{\mathrm{M}}{p}$.

Now M is the moment due to both vertical and horizontal forces acting on the severed portion of the rib; it may therefore be divided into two parts, the moment of the vertical forces acting on the severed part of the rib and the moment of the horizontal component of the reaction. If therefore M^1 is the moment about the centre of all vertical forces, we have—

$$\mathrm{M} = \mathrm{M}^1 - \mathrm{H}\,\eta,$$

and the stress in the member

$$= \frac{\mathrm{M}}{p} = \frac{\mathrm{M}^1}{p} - \frac{\mathrm{H}\,\eta}{p}.$$

The vertical reaction at A due to Q

$$= \mathrm{Q} \times \frac{\mathrm{L} - x}{\mathrm{L}},$$

and if (1) the load Q acts on the severed part of the rib

$$M^1 = Q\frac{L-x}{L}\xi - Q(\xi - x) = \frac{Q}{L}(L\xi - x\xi - L\xi + Lx) = \frac{Q}{L}x(L-\xi),$$

and (2) if Q acts to the left of the portion of the rib severed by the section $x x^1$, $M^1 = \frac{Q(L-x)\xi}{L}$.

Therefore the stress due to the vertical forces

$$= \frac{M^1}{p} = \frac{Q(L-\xi)}{p} \times \frac{x}{L} \text{ or } \frac{Q\xi}{p} \times \frac{L-x}{L},$$

according as the load acts on the severed portion of the rib or to the left of it.

As already mentioned, the stress due to the horizontal force, *i.e.* the horizontal component of the reaction, $= \frac{H\eta}{p}$; and it has been proved that $H = \frac{Qx}{2D}$, where x is measured from the nearest support to Q.

Thus the stress due to the horizontal force

$$= \frac{Q\eta L}{4pD} \times \frac{2x}{L}.$$

If, therefore, in Fig. 114, where A B represents the span of the arch, a perpendicular B b be erected at B $= \frac{Q(L-\xi)}{p}$ and join A b, the ordinate to this line at any point distant x from A $=$ $\frac{Q(L-\xi)}{p} \times \frac{x}{L}$—*i.e.* it equals the stress in the member in question due to the vertical forces for a load at any panel point distant x from A between A and the panel point e on the right of the section $x x^1$.

Now let a perpendicular A a be erected at A $= \frac{Q\xi}{p}$ and join B a, the ordinate at any point to this line $= \frac{Q\xi}{p} \times \frac{L-x}{L}$—*i.e.* it equals the stress due to the vertical forces for a load at any ordinate from B up to the panel point c to the left of the section $x x^1$. Therefore the ordinates to the line A e^1 represent the stress in the given member due to vertical forces, for a load Q at the ordinate and ordinates to B c^1 represent the stress in the member due to vertical forces for a load Q at the ordinate. Since as the load moves from e to c, if x^1 be its distance from c and b the panel length, $\frac{x^1}{b}$ Q is the portion of the load transferred to

and $\frac{b - x^1}{b}$ Q the portion transferred to c, therefore for the load Q at a point between e and c, the stress in the member due to vertical forces is $e\,e^1 \times \frac{x^1}{b} + c\,c^1 \times \frac{b - x^1}{b}$; when $x^1 = b$ it equals $e\,e^1$, and when $x^1 = 0$ it is $c\,c^1$, and it is a linear expression, so we must join $c^1\,e^1$ by a straight line. For a chord member ξ, η are the co-ordinates of a panel point, and c^1 and e^1 coincide, but in other cases they do not. The ordinates at A and B will be drawn upwards or downwards, according to the sign of ξ and $L - \xi$ respectively. From the ordinates to A $e^1\,c^1$ B must be subtracted algebraically the stress due to the horizontal reaction H for a load Q at each panel point consecutively. If we erect an ordinate C C^1 (Fig. 114) at C $= \frac{Q\,\eta\,L}{4\,p\,D}$ and join C^1 to A and B, the ordinate at any point $= \frac{Q\,\eta\,L}{4\,p\,D} \times \frac{2\,x}{L}$, it being remembered that in this case x is measured from the support nearest to the load. The ordinates between the two broken lines A $e^1\,c^1$ B and A C^1 B therefore give the stress in the member due to a load at the position of the ordinate considered, and a further diagram (Fig. 115) may be drawn whose ordinates represent these differences. Adding together the ordinates at the panel points in the last diagram, where they are positive and where they are negative, we get the maximum compression and tension in the member due to any arrangement of the live load.

The case of the centre of moments at infinity has already been dealt with, the ratios of $\frac{\xi}{p}$ and $\frac{\eta}{p}$ having been determined for such a contingency, therefore the influence diagram can also be readily drawn in this case.

It will be observed that change of sign of the moment (or of the stress) will not happen when the load is at a panel point; this is shown in Fig. 113 for the chord members; and for a web member the change must take place in the panel where the section is taken, because when the load is, say, between e and c in the same figure, part of it is transferred by the stringers to e and part to c. Therefore, when the moment about (ξ, η) of the portion of the load Q which is transferred to e, together with that of its reaction at A, equals the moment of the reaction at A of the part of Q transferred to c, the moment will be zero.

It will be noticed that in Figs. 110 and 113 a braced-rib type

of arch is indicated. This type of arch carries vertical supports on which the level platform rests, above or below the crown, consequently the supports and platform do not form an integral part of the arch—that is to say, they do not relieve the arch of any stress, nor tend to stiffen it, nor do they assist in the erection. If, however, the top member of the rib is made horizontal, the platform can be carried upon it directly, and each half of the bridge can be built out as a cantilever by simply tying back the ends of the horizontal member to secure anchorages. The design is then known as a spandrel-braced arch. The method of calculation is exactly the same as indicated for the braced rib, and since the centre lines of the chord members in any bay meet at a finite distance any question of the centre of moments being at infinity is eliminated.

TWO-HINGED ARCH

The disadvantage of the three-hinged arch for railway traffic is its lack of rigidity, excessive longitudinal vibration being sometimes developed under traffic at even moderate speeds. This drawback is to a great extent removed by the omission of the centre hinge in either the braced-rib or spandrel-braced type of arch, but the hinges at the points of support are retained.

The calculation of the stresses in this case is more tedious than for the three-hinged arch, owing to the fact that the horizontal component of stress cannot be obtained in so direct a manner; but, as already explained, the elastic properties of the members have to be investigated in order to obtain the necessary equation for it.

Rib with Solid Web.—The vertical component of the reactions can be found as before, by taking moments about either hinge, because the horizontal component acts through the hinges. Before the stresses in the members for any given load can be found, it is necessary to obtain the horizontal component of the reactions due to the load. For this purpose the effect is considered of the bending of each element of length of the arch in tending to move

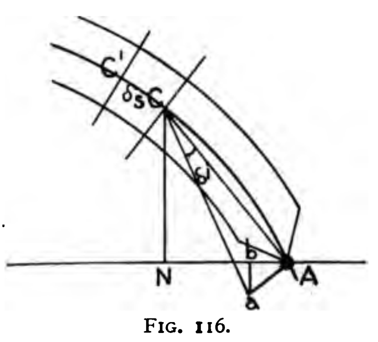

FIG. 116.

one point of support, supposing the other one to remain immovable. Suppose the arch to be divided up along the centre line into lengths δs, and let $C C^1$ (Fig. 116) be one of these elements of length. Owing to the bending of this element, CA will tend to be deflected downwards through an angle δi, where δi is the angle turned through by the section at C at one end of δs relatively to the section at C^1 at the other end of δs. This bending would cause A to move to a and the horizontal projection of this displacement is $A b = \delta x$. Now since the span remains unaltered in length the sum of all the movements $A b$ must be zero—*i.e.* $\Sigma A b = 0$. It is also clear that since the height of the abutments remains constant that $\Sigma a b$ must also be zero.

Let x, y be the co-ordinates of C.

By similar triangles we have

$$\frac{A b}{A a} = \frac{C N}{A C} \quad \text{or} \quad \delta x = y \frac{A a}{A C}.$$

But $A a = A C \delta i$. $\therefore$ $\delta x = y \delta i$; similarly $a b = \delta y = x \delta i$.

From the property of elastic deflection of a beam we have $\delta i = \frac{M \delta s}{E I}$, where M is the resultant bending moment on the strip, E is Young's modulus of elasticity, and I the moment of inertia of the cross-section;

$$\therefore \quad \delta x = \frac{M y \delta s}{E I}, \quad \text{and} \quad \delta y = \frac{M x \delta s}{E I}.$$

By the conditions above quoted $\Sigma \delta x = 0$ and $\Sigma \delta y = 0$;

$$\therefore \quad \Sigma \frac{M y \delta s}{I} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\Sigma \frac{M x \delta s}{I} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Now M is equal to the moment about C of the external vertical and horizontal forces, and may be written

$$M = M^1 - H y,$$

where M^1 is the moment of any loads and the vertical reaction to the right of C, and $-H y$ is the moment about C of the horizontal reaction H at the right hinge.

Substituting this value for M in (1) and (2),

$$\Sigma \frac{M^1 y \delta s}{I} - H \Sigma \frac{y^2 \delta s}{I} = 0, \text{ and } \Sigma \frac{M^1 x \delta s}{I} - H \Sigma \frac{x y \delta s}{I} = 0;$$

$$\therefore \quad H = \frac{\Sigma \frac{M^1 y \delta s}{I}}{\Sigma \frac{y^2 \delta s}{I}} \quad \text{or } H = \frac{\Sigma \frac{M^1 x \delta s}{I}}{\Sigma \frac{x y \delta s}{I}}$$

Thus H may be found very approximately for a given load by dividing the centre line into a number of equal parts δs, in which case δs cancels out of the expressions, the value of M^1 and I if variable must be calculated for each section and y measured. It will be observed that in calculating the deflection no allowance has been made for that due to thrust along the axis nor for that due to shear, but these are small compared with that due to bending moment, owing to the leverage in the latter case magnifying the effect to so large an extent, as shown in Fig. 116, where the bending of δs is magnified by the leverage C A.

If I be assumed to vary as sec θ, where θ is the inclination of the axis at the section under consideration, then I equals the moment of inertia at the centre section multiplied by sec θ, and since $\delta s = \delta x \sec \theta$ the formulas simplify, and the expressions become—

$$H = \frac{\Sigma M^1 y \, \delta x}{\Sigma y^2 \, \delta x} \quad \text{and } H = \frac{\Sigma M^1 x \, \delta x}{\Sigma x y \, \delta x}.$$

It will be seen from the chapter on deflections that $\int \frac{M^1 y \, d s}{E I} (= u)$ is the deflection of the right hinge horizontally under the bending moment M^1, therefore $u = H \int \frac{y^2 ds}{E I}$, which is evident when it is remembered that the force H acting inwards at the hinge counteracts the outward deflection due to the load, or, in other words, if the load produces the deflection u outwards, H acting inwards produces the deflection u inwards, and by the formula for deflection, since the moment at the axis of a horizontal force H at the hinge $= - H y$, the horizontal deflection at the hinge due to a horizontal force H acting there $= - \int \frac{H y . y \, d s}{E I}$ or $- H \int \frac{y^2 \, d s}{E I}$. Therefore by equating the deflection due to the load to that due to the horizontal force H we get $\int \frac{M^1 y \, d s}{E I} = H \int \frac{y^2 d s}{E I}$, the same result as before.

In the case of the parabolic rib H can be integrated out from the formula, but for the circle the work of integration is tedious.

Dead Load Stresses.—When H has been thus calculated for the dead load, the stresses in the arch may be obtained by drawing the equilibrium polygon. Since the tangent to the latter at any section is the direction of the resultant force acting at that section, whose amount is given by the corresponding

ray in the force diagram, by resolving this along the tangent to the axis at the section the axial thrust is obtained, and by resolving it perpendicularly to it the shear is obtained; and the resultant bending moment equals this force multiplied by the perpendicular upon it from the axis at the section, or what is the same thing, H × the difference between the ordinates of the equilibrium polygon and the axis at the section. The bending moment increases the compression in the upper flange when positive and in the lower flange when negative.

Live Load Stresses.—To find the positions of the live load which give the maximum bending moment, thrust, and shearing force. Consider a single load Q as before. The vertical reactions can be determined in the same manner as for the three-hinged arch and the horizontal component as explained above. With a single load the resultant reactions must for equilibrium meet on the line of action of the load, and for a series of loads they must meet on their resultant.

Bending Moment.—If, therefore, the vertical reactions and the value of H be determined for a series of positions of Q, the

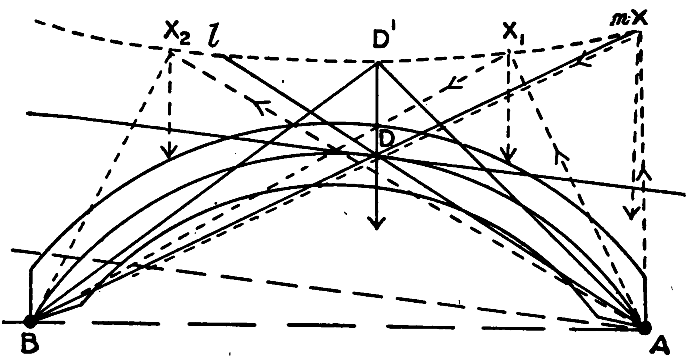

FIG. 117.

point of intersection of the resultant reactions can be found for each position and the locus of their points of intersection drawn. It can then be readily ascertained which positions of the load Q give a positive bending moment at any point D (Fig. 117), and which positions give a negative bending moment. Consider the forces acting on the portion of the rib A D. Produce A D and B D to l and m respectively. A load to the right of m, together with its reaction at A, will have a resultant X B

along the line of the reaction at B, which necessarily meets that at A at the point X on the reaction locus; the moment of the resultant along X B, which is to the right of D, is negative or clockwise. Between m and D^1 the resultant is still along the varying position of X_1 B, which is then to the left of D, and therefore gives a positive moment, when the load comes to D^1 it only affects the portion of the rib A D by the reaction it causes at A; the reaction at A for a load between D^1 and l continues to give a positive moment. For a load at l the reaction at A passes through D, therefore its moment about D becomes zero, and for points to the left of l it has a negative value. Thus for loads from A to m and from l to B the bending moment at D is negative, and for loads from m to l it is positive. Therefore the maximum positive moment at D is when the length l m only is covered, and the maximum negative moment when the lengths m A and l B only are covered.

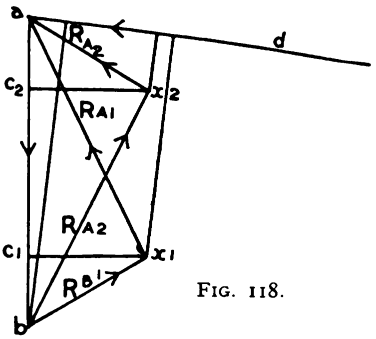

FIG. 118.

This bending moment will increase the compression in the upper flange when positive, and in the lower flange when negative.

Axial Thrust.—To find the thrust along the axis at any point D due to load at all the panel points, it is necessary to find V_A, V_B, and H for a load at each, then draw the force diagram for each load. If, therefore, in Fig. 118 b a represents Q for the position X_1, a c_1 and c_1 b represent the corresponding vertical reactions at A and B respectively and a x_1 and b x_1 are the resultant reactions. The same letters with the suffix 2 correspond to the position X_2 of the load. For loads to the right of D the resultant of the load and the reaction at A is equal and opposite to the reaction at B. Therefore for Q at X_1 the thrust will be $-R_{B1}$ resolved along a d drawn parallel to the tangent at D; loads to the left of D only affect the length A D owing to their reaction at A; therefore for the position X_2 of Q the thrust will be R_{A2} resolved along a d.

If it be remembered that A X_1 B and A X_2 B are the equilibrium polygons for the positions X_1 and X_2 of the load respec-

tively, it is obvious that the lines $X_1 B$ and $A X_2$ respectively represent the line of the total thrust at the vertical section through D, the reasoning is shortened.

It is obvious that the loads at all the panel points increase the thrust at D; but the maximum compression due to bending moment does not occur with the same disposition of load as the maximum thrust, as the former has been shown to occur with only part of the span loaded. If moments be taken about the centres of moments of the flanges instead of about the axis, the maximum stresses in the flanges would be obtained, as shown in the case of the three-hinged arch.

Shearing Force.—To find the maximum shear at D (Fig. 117). For loads to the right of D the resultant of the load and its reaction at A being downwards along the line of reaction at B, the shear at D is downwards and therefore negative; for loads to the left of D only the reaction at A affects the portion of rib A D. This would obviously give an upward or positive shear at D, until the load reaches such a position that the reaction at A becomes parallel to the tangent at D, unless the load reaches B before this happens. If such is not the case to the left of the point where a line through A parallel to the tangent at D meets the reaction locus, the shear will obviously again become negative. Thus the maximum positive shear will be when the span is loaded only between D^1 and the point where a line from A parallel to the tangent at D meets the reaction locus, or from D to B, as the case may be; and the maximum negative value occurs when the portions outside the above only are loaded.

Arch with Braced Web.—In this case the formula for H already obtained does not strictly apply, and a more correct value may be deduced by utilising the formula for the deflection at a point of a braced girder.

When a single load Q is applied to the girder, if one end be capable of moving horizontally on rollers the structure would be statically determinate, and the stress S^1 in any member length L can be calculated by drawing the stress diagram. The stress s in each member, due to a horizontal force equal unity applied at the free end, can be similarly found; and the horizontal deflection at this free end due to the load Q can therefore be calculated from the formula in question. The horizontal deflection therefore $= u = \Sigma \frac{S^1 s L}{A E}$, where A is the cross-sectional area of the member length L. Now th[illegible]tal component

of the reaction, H, deflects the point of support back to its fixed position, and if H be taken as the acting force $-$ H s would be the stress in the member length L due to it, since s is the stress in this member due to unit horizontal force acting outwards; therefore the deflection due to H acting inwards

$$= -u = -\Sigma \frac{s\,\mathrm{H}\,s\,\mathrm{L}}{\mathrm{A\,E}} = -\mathrm{H}\,\Sigma \frac{s^2\,\mathrm{L}}{\mathrm{A\,E}}.$$

Equating these two expressions for u—

$$\Sigma \frac{\mathrm{S}^1\,s\,\mathrm{L}}{\mathrm{A\,E}} = \mathrm{H}\,\Sigma \frac{s^2\,\mathrm{L}}{\mathrm{A\,E}} \text{ or } \mathrm{H} = \frac{\Sigma \frac{\mathrm{S}^1\,s\,\mathrm{L}}{\mathrm{A}}}{\Sigma \frac{s^2\,\mathrm{L}}{\mathrm{A}}} \quad . \quad . \quad . \quad (1)$$

Of course A is not known, and an approximate value has to be assumed in the first instance, and having thus obtained a preliminary value for H the maximum stress in any member may be obtained by drawing its influence diagram, giving the stress in the member due to a load at each panel point taken separately. Then by adding all the stresses of the same sign at the panel points, the maximum compression and tension in the member are obtained.

For this purpose a section is taken intersecting the member in question and two others, and the centre of moments is the intersection of the two other members, produced if necessary; let ξ, η be the co-ordinates of this point with the right hinge A as origin and the line joining the hinges A and B as axis of x, and let p be the stress perpendicular from the centre of moments on to the member in question, and M be the moment about the centre of all the external forces acting on the severed portion of the arch. Then the stress in the member for the actual load $= \mathrm{S} = \frac{\mathrm{M}}{p}$.

Now $\mathrm{M} =$ the moment M^1 of the vertical loads and their reaction at the hinge and that of the horizontal reaction H—*i.e.* $\mathrm{M} = \mathrm{M}^1 - \mathrm{H}\,y$—

$$\therefore \qquad \mathrm{S} = \frac{\mathrm{M}}{p} = \frac{\mathrm{M}^1}{p} - \mathrm{H}\frac{y}{p}.$$

As in the case of the three-hinged arch, if Q is at a distance x from A, the vertical reaction at $\mathrm{A} = \mathrm{Q} \times \frac{\mathrm{L}-x}{\mathrm{L}}$.

If (1) the load Q acts on the severed portion of the rib to the right of the section taken—

$$\mathrm{M}^1 = \mathrm{Q}\frac{\mathrm{L}-x}{\mathrm{L}}\,\xi - \mathrm{Q}\,(\xi - x) = \frac{\mathrm{Q}}{\mathrm{L}}\,x\,(\mathrm{L}-\xi).$$

(2) If Q acts to the left of the section, $M^1 = \frac{Q}{L}(L - x)\,\xi$.

The stress due to the vertical forces equals the above divided by p. To obtain a diagram of the stress in the member due to a load at any panel point, as in Fig. 114, an ordinate B b $= Q\frac{(L-\xi)}{p}$ is drawn, and its extremity joined to A; similarly an ordinate A a $= \frac{Q\xi}{p}$ is drawn at A and its extremity joined to B; the ordinates of the portions of these lines to left and right of the panel in question represent the stresses, for vertical loads, in the member due to a load Q at the ordinate. In the case of a chord member the two lines meet at the panel point, which is the centre of moments; but in the case of web members the ordinates at the extremities of the panel are joined by a straight line for the same reason as before.

The difference comes in when the line whose ordinates represent the stress due to the horizontal reaction H is considered. For the three-hinged arch H for the load Q at any panel point was simply proportional to the distance from the nearest hinge; but now it varies in a more complicated manner, and its value for a load at each panel point has to be calculated by aid of equation (1), page 200. The result in each case is multiplied by $\frac{y}{p}$ and plotted at the proper panel point; instead of being two straight lines (as in Fig. 114) meeting at the centre, it is a curve. A further diagram, similar to Fig. 115, gives the algebraic difference of the two sets of ordinates, and represents the stress in the member for a load at each ordinate, from which the maximum compression and tension in the member can be found by addition of the plus and minus ordinates at the panel points.

Effect of the Shortening of the Rib on the Value of H

If f be the average intensity of thrust through the rib, the shortening

$$= \int_0^L \frac{f\,dx}{E} = \frac{f\,L}{E}.$$

This shortening will tend to produce an inward pull H_s. Now

the deflection due to H_S acting inwards $= -H_S\int\frac{y^2dx}{EI}$;

$$\therefore \frac{fL}{E} = -H_S\int_0^L \frac{y^2dx}{EI}.$$

If the rib is parabolic $y = \frac{4D}{L^2}x(L-x)$, where D is the rise of the arch. Substituting this value and integrating, we get—

$$\frac{fL}{E} = -\frac{8}{15}\frac{H_S D^2 L}{EI},$$

or

$$H_S = -\frac{15}{8}\frac{fI}{D^2}$$

where I is the moment of inertia at the centre.

Effect of Temperature Variation on the Value of H

If α be the co-efficient of linear expansion and T the range of temperature above or below the average, the alteration in the span if one end were free to move would be $\pm\alpha TL$. If H_T be the horizontal thrust caused by this change of temperature the horizontal deflection due to this force $= H_T\int\frac{y^2dx}{EI} = \pm\alpha TL$

$$\therefore H_T = \pm\frac{15}{8}\frac{EI\alpha T}{D^2}.$$

ARCHES WITH FIXED ENDS AND NO HINGE

If a load Q be applied at any point distant x from A, it is necessary for equilibrium (see Fig. 119) that the sum of the

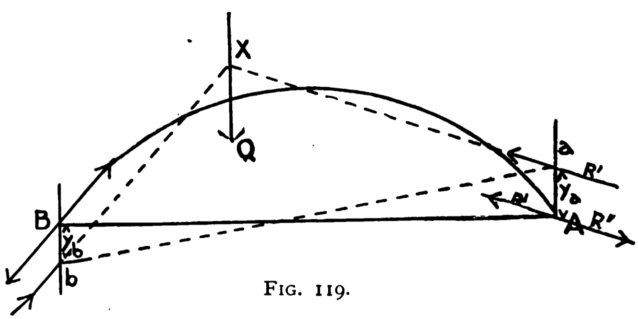

FIG. 119.

vertical components of the reactions is equal to Q, and that the moment about either end of the reaction at the other end must

of the load equals the moment of the horizontal com-
ɜnts of the reactions, which do not in this case act in the ɪe line. Expressing these conditions in the form of equations—

$$Q = V_A + V_B \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$V_B \times L + Q\,x + H\,(y_a - y_b) = 0 \;. \quad . \quad (2)$$

The second equation shows that V_A and V_B are not inversely their distances from Q, as in the previous cases.

Fig. 119 shows that the closing line *a b* of the equilibrium ɔlygon for the load Q and its reactions is in this case an inɩined line.

It will be seen that there are five unknown quantities in the ɩbove equations, consequently three more equations are required to determine these. Two equations are obtained in the same manner as for the rib hinged at the two ends, namely—

$$\Sigma \frac{M\,x\,\delta s}{I} = 0 \quad \text{and} \quad \Sigma \frac{M\,y\,\delta s}{I} = 0.$$

Since M equals the moment of the loads and vertical reactions M′, and the moment of the horizontal reactions M″, we have

$$M = M' + M''$$

Substituting for M in the above we get

$$\int \frac{M'\,x\,d\,s}{I} + \int \frac{M''\,x\,d\,s}{I} = 0 \quad . \quad . \quad (3)$$

and

$$\int \frac{M'\,y\,d\,s}{I} + \int \frac{M''\,y\,d\,s}{I} = 0 \quad . \quad . \quad (4)$$

A further equation can be obtained from the fact that, since the ends of the arch are fixed, the inclination of the axis at the two ends remains unaltered when the load is applied; consequently, if the change of inclination of the tangent to the centre line be integrated from one end of the arch to the other, the total change is zero. Expressed as an equation—

$$\int_0^L d\,i = 0,$$

or substituting for *d i*

$$\int_0^L \frac{M\,d\,s}{E\,I} = 0, \text{ or } \int_0^L \frac{M'\,d\,s}{I} + \int_0^L \frac{M''\,d\,s}{I} = 0 \quad .$$

If I be supposed to vary as sec θ, since $d\,s = d\,x \sec\theta$, we r write *d x* for *d s* in these equations, I then being the mon of inertia of the central section.

For a parabolic rib these equations can be ꜰ

locus of intersection of the reactions obtained, but as in this case they do not pass through any definite point of support, it is necessary also to find the curve which is the envelope of the lines of reaction in order to permit of the reactions being drawn for any position of the load. The methods already considered may be applied to determine the stresses.

The influence of the shortening of the rib and of temperature changes in affecting the horizontal thrust is much more marked in this case.

Masonry Arch.—The masonry or brick arch, as usually built without hinges, is an example of the third type, but as the modulus of elasticity of masonry and mortar is very variable, and as a very slight accommodation of the latter in the joints near the springing may cause the arch to approximate much more to one hinged at the supports, it is not usual to apply the above methods in this case. Another feature of the masonry arch is that the dead load is great compared with the live load. In order to avoid tension in the masonry it is necessary that the line of pressure shall not deviate from the centre line of the arch more than about one-sixth of the thickness. The reason for this is easily seen, for if A B (Fig. 120) be the length b of a joint in the arch, and P be the normal component of the resultant pressure acting at D distant x from the centre C, then we have P acting at D is equivalent to P acting at C and the couple $P \times x$. If we consider unit width of the joint, since the length A B equals b, the compression due to P at the centre $= \frac{P}{b}$.

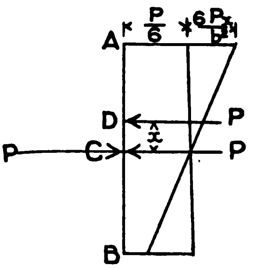

FIG. 120.

The maximum tensile and compressive stress due to the couple $= f$ say, then $\frac{2f}{b} = \frac{12 P \times x}{b^3}$ $\left(\text{from the formula } \frac{f}{y} = \frac{M}{I}\right)$.

$\therefore \; f = \frac{6 P x}{b^2}$, and this must not exceed the uniform compression $\frac{P}{b}$ due to P acting at the centre, or else the tension caused by the couple would be greater than the latter and a tension would result. $\therefore$ the limiting value is given by

$$\frac{6 P x}{b^2} = \frac{P}{b} \text{ or } x = \frac{b}{6},$$

thus P must be within the middle third of the joint or there will be tension at the extremity of the joint furthest from P.

If the line of pressure is at the centre of the joint the maximum intensity of pressure is of course equal to the average intensity of pressure $= \frac{P}{b}$. When the line of pressure is at a distance $\frac{b}{6}$ from the centre, as just proved, the maximum intensity of pressure $= \frac{2P}{b}$. For other distances the maximum intensity of pressure $= \frac{P}{b}\left(1 + \frac{6x}{b}\right)$, where x is the distance of the line of pressure from the centre. If x is $> \frac{1}{6}b$, the maximum tension $= \frac{P}{b}\left(\frac{6x}{b} - 1\right)$. If the line of pressure deviates more than one-sixth the length of the joint from the centre—*i.e.* if it is outside the middle third—it does not necessarily follow that the structure will be endangered, but the end of the joint farther away from the line of pressure will be in tension, and will in course of time probably open for such a length that tension no longer exists; the effect of the opening of the joint under the tension will be to further increase the maximum intensity of compression, as will be obvious from the following considerations:—

Suppose the line of pressure cuts a joint at one-quarter the length of the joint from one end of it, its distance from the centre $= x = \frac{b}{2} - \frac{b}{4} = \frac{b}{4}$. Before the joint opens the maximum intensity of compression $= \frac{P}{b}\left(1 + \frac{6}{4}\right) = 2\frac{1}{2}\,\frac{P}{b}$, and the maximum intensity of tension $= \frac{P}{b}\left(\frac{6}{4} - 1\right) = \frac{P}{2b}$. When the joint opens to such a point that tension no longer exists, the length of joint in compression equals three times the distance of the line of pressure from the end of the joint $= 3 \times \frac{b}{4}$ —*i.e.* the length of the crack $= \frac{b}{4}$.

The maximum intensity of compression now equals

$$2 \times \frac{P}{\frac{3}{4}b} = \frac{8}{3}\,\frac{P}{b} = 2\frac{2}{3}\,\frac{P}{b},$$

which is greater than its value $2\frac{1}{2}\,\frac{P}{b}$ before the crack occurs.

Again, if the line of pressure cuts a joint at one-fifth the length of the joint from the end, its distance from the centre $= x = \frac{b}{2} - \frac{b}{5} = \frac{3b}{10}$. Before the joint cracks, the maximum intensity of compression $= \frac{P}{b}\left(1 + \frac{9}{5}\right) = 2\frac{4}{5}\frac{P}{b}$, and at the same time the maximum intensity of tension $= \frac{P}{b}\left(\frac{9}{5} - 1\right) = \frac{4}{5}\frac{P}{b}$. When the joint opens to such a point that tension no longer exists, the length of joint in compression equals three times the distance of the line of pressure from the end of the joint $= 3 \times \frac{b}{5}$, and the length of the crack will be $\frac{2}{5}b$.

The maximum intensity of compression then equals

$$2 \times \frac{P}{\frac{3b}{5}} = \frac{10}{3}\frac{P}{b} = 3\frac{1}{3}\frac{P}{b},$$

which is greater than its value $2\frac{4}{5}\frac{P}{b}$ before the joint opens, and as the line of pressure recedes from the centre this difference becomes more marked.

It is thus seen that the maximum intensity of compression increases rapidly above its mean value of $\frac{P}{b}$ when the line of pressure is at the centre, as that line recedes from the centre, and moreover the thickness of the voussoirs for the length that the joints open is wasted so far as resisting the loads is concerned. In the case of a concrete arch this loss may be partly avoided—*i.e.* to the extent of the above difference between the value of the intensity of compression before and after cracking—by reinforcing the concrete with steel rods at the parts where tension will occur, so that the tension may be resisted without any cracking taking place.

The object aimed at in designing a masonry, brick, or concrete arch is to avoid any tension, and the consequent increased value of the intensity of compression, by causing the line of pressures to lie within the middle third of the thickness of the arch, if possible, by suitably regulating the shape of the arch and the thickness of the voussoirs.

It is necessary, in the first place, to make some preliminary estimate of the necessary thickness of the arch at the crown, and the formula generally made use of is :—

The preliminary thickness of the arch at the crown in feet = $0{\cdot}4\sqrt{r}$, where r is the radius of curvature at the crown in et. It is generally desirable to increase the thickness towards e abutments.

Next, the elevation of the arch of the given span and rise and

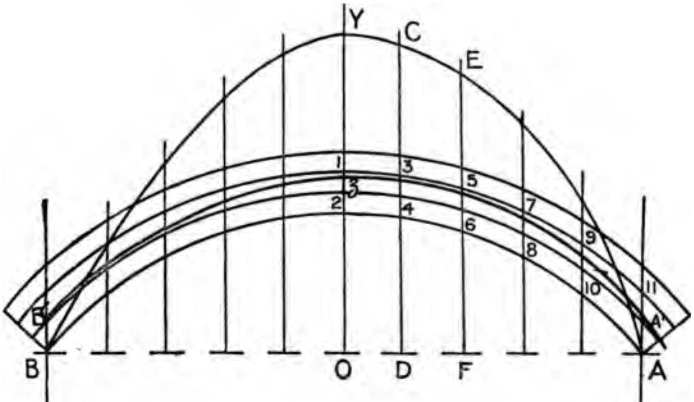

FIG. 121.

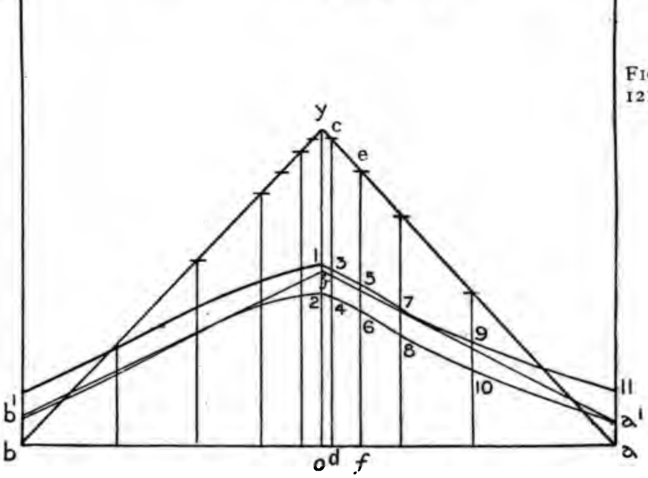

FIG. 122.

f the preliminary shape and thickness is drawn (Fig. 121), and he thickness divided into three equal parts at each section, nd the lines, representing one-third the thickness of the voussoirs from each extremity, drawn. The lines of the abutments re extended upwards to intersect the lower one-third line.

On the base A B the bending-moment diagram of the acting

vertical loads is drawn. In order to draw this the arch between A B is divided into a convenient number of equal parts, and the weights between each consecutive pair of vertical sections of the arch, spandrel filling, roadway, parapet walls, and load determined, which enables the bending-moment diagram for the vertical forces to be plotted. It is necessary to consider both the case (1) of the load covering the whole of the span and (2) when the load covers only one-half the span from one end. Now the ordinates of the equilibrium polygon or line of pressures (or linear arch, as it is sometimes called) are known to be proportional to those of the above bending-moment diagram, and it is necessary to find the ratio by which the ordinates of the latter must be multiplied in order to give those of the former. We have already seen that in order to avoid tension the equilibrium polygon must lie within the middle third of the arch, therefore this is the criterion which must be satisfied by altering the thickness of the arch or its shape until it is arrived at. To see whether the preliminary arch taken satisfies this condition in Fig. 122 draw $o\,y$ equal to the centre ordinate of the bending-moment diagram in Fig. 121, join $a\,y$ and $b\,y$. On $a\,y$ mark off the ordinates $c\,d$, $e\,f$ . . equal to the ordinates C D, E F . . in Fig. 121. On $o\,y$, $c\,d$, $e\,f$ . . . mark off 01, 02 ; $d3$, $d4$; $f5$, $f6$; . . equal to the corresponding ordinates in Fig. 121, and giving the position of the extremities of the middle thirds on the various ordinates, draw a curve through the points 1, 3, 5, . . . 11, and through the points 2, 4, 6, . . . a^1. It is then obvious that if a straight line can be drawn between these two curves without intersecting them, and the points where it cuts the ordinates be transferred to the corresponding ordinates in Fig. 121, that a curve drawn through these latter points will fall in the middle third of the arch. When the load covers only one-half of the span from one end the bending-moment diagram will not be symmetrical, therefore the same proceeding must be gone through for the other half of the span $o\,b$. Then if the arch is of suitable shape and thickness, it will be possible from some point z between 1 and 2 in Fig. 122 to draw straight lines between the curves representing the middle third of the arch. In case it is possible to draw a number of such pair of lines, the pair which make the largest angles with $a\,b$ are taken, the angle being steeper at the abutment where the reaction is greatest, since H is constant. Suppose this pair of lines to be $z\,a^1$ and $z\,b^1$. Then $a^1\,b^1$, and the corresponding line $A^1\,B^1$ in Fig. 121, are the bases

which the bending-moment diagram of the vertical forces ould be plotted, where A $A^1 = a\,a^1$ and B $B^1 = b\,b^1$. Calling e new position of y, y^1, it will be seen that the ordinates of z are proportional to those of $a^1\,y^1$—*i.e.* to the bending oments of the vertical forces at the various sections; and if e length of the ordinates of $a^1\,z$ be transferred to the corres- onding ordinates in Fig. 121, a curve $A^1\,Z\,B^1$ lying within the iddle third will be obtained, which will be the line of pressure equired; for it is evidently a possible line of pressure, as its rdinates are proportional to the bending moment of the vertical forces, and it is assumed that of the possible lines of pressure that will be the actual one which causes the thrust required to be exerted by the abutment on the arch to be a minimum. By making the line $a^1\,z$, the steepest possible, on the more heavily loaded side, that line of pressure has been obtained which has the greatest possible inclination to the horizontal at the abutments—*i.e.* the ratio of the horizontal component f the pressure on the abutments to the vertical component s the smallest possible. Now the vertical component of the hrust at the abutments is fixed by the load, therefore the line f pressure has been found which causes the minimum resultant hrusts on the abutments, and consequently the minimum re- ctions of the abutments on the arch. If, therefore, this line f pressures falls within the middle third, the condition that here shall be no tension is also satisfied, but if not the shape or hickness of the arch, or both, must be altered till this result s effected; or until the deviation of the line of pressure from he centre is not greater than some pre-determined amount. When the line of pressure satisfying the above conditions has een determined, it only remains to be seen whether the maximum ntensity of compression and shear is excessive at any section; he former may vary from 6 to 20 tons per square foot and the tter from 2 to 7 tons per square foot, according to the character f the concrete or masonry of which the arch is to be built. The ays of the force diagram for the final equilibrium polygon or line f pressures gives the total thrust at each section, and the tangent o the equilibrium polygon at any joint gives the inclination nd position of the resultant thrust at that joint. Its normal omponent P and its component S parallel to the joint can herefore be at once obtained; if, as before, b is the length f the joint the maximum intensity of compression is $\frac{P}{b}\left(1 + \frac{6\,x}{b}\right)$,

where x is the distance of the point of application of the resultant thrust from the centre of the joint. The intensity of shear $= \frac{S}{b}$. If either of these are too great the thickness of the arch will have to be increased. If the excessive intensity only occurs at the haunches, the thickness may be increased at the springing and left the same as before at the crown.

When an arch whose soffit is the segment of a circle is overloaded by a load whose bending-moment diagram approximates to a parabola, since the radius of curvature of the latter is

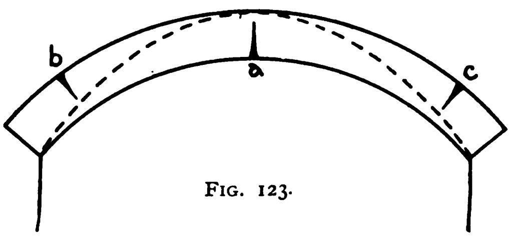

FIG. 123.

least at the vertex and gradually increases away from that point, it is obvious that the line of pressure will approach the outside of the arch at the crown and fall near to the soffit in the haunches, as indicated in Fig. 123. The portions of the arch marked a, b, c, in the figure will therefore come into tension, and failure would take place by the arch cracking at such sections; unless it is a concrete arch reinforced by steel rods in the parts where tension exists. In the latter case, if overloaded, the arch might fail owing to the great increase in the maximum intensity of compression at these sections, or by shearing.

Abutments.—To design the abutment for an arch when the line of pressures has been found, it is necessary to know accurately the amount and direction of the maximum thrust on the abutment. If a point on the line of pressure be taken, say, on the vertical centre line of the arch, and if D is the vertical depth from that point to the closing line of the equilibrium polygon, and if H be the horizontal component of the force along the closing line, which would also be the horizontal component of the thrust on the abutments, then $H \times D =$ the bending moment of the vertical forces at the centre. This gives H the horizontal component of the thrust on the abutment. Now V the vertical component $=$ the portion of the weight transferred to that

end of the span, and can be found by taking moments about the other end of the span. Therefore the total thrust

$$\text{R} = \sqrt{\text{H}^2 + \text{V}^2},$$

and the tangent of its inclination to the horizontal $= \frac{\text{V}}{\text{H}}$.

In Fig. 124 R represents the position of this thrust. Let O be the point where this thrust intersects the line of action of W, the weight above the horizontal section through O to the

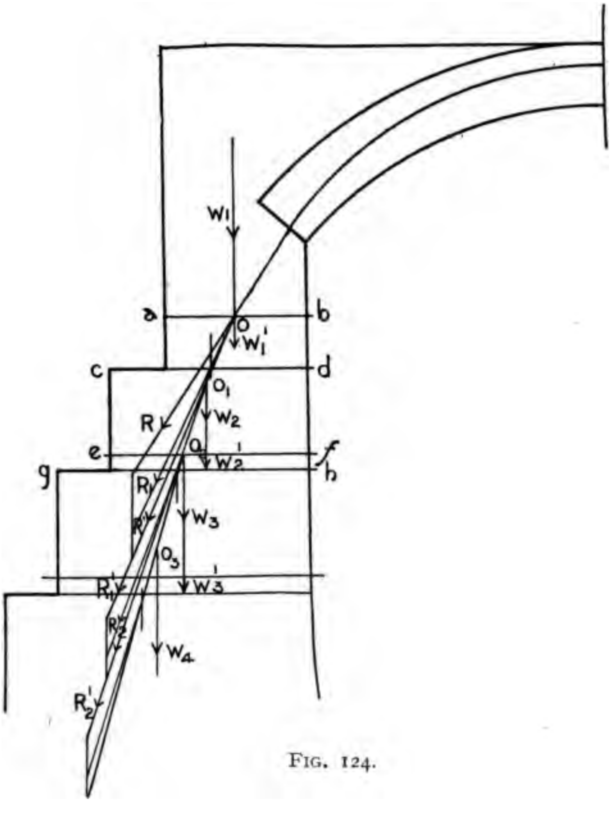

FIG. 124.

top of the filling in the spandril of the arch. From O draw R in magnitude and direction, and from its extremity plot W_1, the resultant of these two $= \text{R}_1$, and is the resultant force on the horizontal section $a\,b$. R_1 intersects the vertical through the point distant $\frac{a\,b}{3}$ from a on the horizontal line $c\,d$,

and let W_1^1 be the additional weight between this section *c d* and that at *a b*. From the extremity of W_1 plot W_1^1, then the resultant R_1^1 of R_1, W_1, and W_1^1 is the resultant force acting on the section *c d*, and it will act just inside the middle third of the section, and therefore tension is avoided. The first offset is made at this level, and a line is drawn through the centre of gravity of the block below *c d*—*i.e.* through the middle point of *c d*. Let R_1^1 meet this line in O_1. From O_1 plot R_1^1 in magnitude and direction, then this line intersects the vertical through a point distant $\frac{c\,d}{3}$ from *c* on the horizontal line *e f*, and let W_2 be the weight between the horizontal line *e f* through this point and *c d*. From the extremity of R_1^1 plot W_2, and let R_2 be the resultant of R_1^1 and W_2. R_2 intersects the vertical through one-third the width of *c d* from *c* on the horizontal section *g h*, and let W_2^1 be the weight of the block between the horizontal sections *e f* and *g h*. At the extremity of W_2 plot W_2^1, then R^1_2, the resultant of R_2, W_2, and W_2^1, will cut the joint *g h* slightly within one-third of the width ; there will therefore be no tension there. The next offset is made at *g h* and the further steps are a repetition of the method of proceeding just described between the offsets at *c d* and at *g h*.

The thickness at *a b* and the magnitude of the offsets must be taken large enough to prevent the abutment increasing in thickness too rapidly.

If there is any possibility of the arch ever having to carry its load without the assistance that may be obtained from the backing behind the abutment, no credit must be taken for the help the earth pressure against it would afford. But if this pressure is certain to be always present, the thickness of the abutment may be reduced. The pressure due to earth will be considered later.

CHAPTER XI

SUSPENSION BRIDGES

If instead of the curved structure corresponding to the curved flange of a bowstring girder in its upright position, it corresponds to the curved flange of an inverted bowstring girder, it is obvious that it will be primarily in tension due to the load, and it will differ from the inverted bowstring girder, due to the fact that instead of the ends being kept apart by the straight flange as part of the structure, an external horizontal force is applied at each point of support to effect this.

It has already been seen that, if the inverted parabolic bowstring girder is uniformly loaded, the parabolic member is in tension and the straight member is of course in compression; but there is no stress in the web members because the vertical component of the stress in the parabolic flange balances the shearing force due to the load. Consequently, if the suspended parabolic member be flexible and is subjected to a load uniformly distributed horizontally, attached to the suspended member in such a manner as to produce the same effect as if continuously applied, the flexible member would of itself be able to support that load, as the only stress produced in it would be a tension directed everywhere along its tangent at the point considered. Thus the flexible member may consist of a series of links pinned together at the points at which the load platform is attached to it, or it may be a flexible wire cable built up of a large number of separate strands, the advantage of the latter form consisting in the fact that the tensile strength of the wire per unit of cross-section is several times as great as that of forged links.

Another way of regarding the question, and one which enables the effect of the load to be realised when it is not uniformly distributed horizontally, is the following:—In a suspension bridge a series of vertical loads and the inclined reactions are in equilibrium, therefore, as we have already seen, the tangent to the equilibrium polygon at every vertical section is the resultant, acting across that section, of all the external forces

on one side of that section, and therefore if the flexible structure coincides with the equilibrium polygon, the acting forces would be balanced by direct tensions along this member. But if the centre line of the structure does not coincide with the equilibrium polygon, the resultant force acting along the equilibrium polygon at the section is equivalent to an equal and parallel force at the centre line of the structure (whose component normal to its cross-section at that point causes a tension in the structure, and whose component in the cross-section causes a shear), and a couple equal to the resultant force acting along the equilibrium polygon multiplied by the perpendicular from the centre line of the structure on that resultant force; in the latter case a flexible member would be caused to alter its shape, until its centre line coincided with the equilibrium polygon.

Again, it has been proved that the ordinates of the equilibrium polygon multiplied by H, the constant horizontal component of the forces acting along the equilibrium polygon, represent the bending moments of the vertical forces acting. Thus, if a flexible member has to support a load uniformly distributed horizontally, its shape would be a parabola whose centre ordinate D, below the line of the supports, would be $\frac{w\,L^2}{8\,H}$, where w is the intensity of the load and L is the span. In case the load on the platform is not a uniformly distributed load, the shape of the flexible suspension member would no longer remain parabolic, but would tend to take the shape of the equilibrium polygon for the actual loads. When, therefore, such a load moves across a flexible suspension bridge, the shape of the latter would continually change and the shape of the roadway would be a continually changing sinuous curve, making it unpleasant and sometimes difficult for a load to be taken across. If the time of oscillation caused by the moving load happened to coincide with the time of oscillation of the structure, its amplitude would increase, and the structure might even be endangered. This is a circumstance which may occur in a gale of wind if the gusts happened to have a periodicity coincident with the natural time of oscillation of the bridge, and, although the chances of such a coincidence are remote, cases have been known where considerable damage has resulted from this cause, even when the bridge was stiffened.

It has been found that if the load is carried primarily on a comparatively light girder, which is hung from the flexible

member by hangers, that the shape of the flexible member is not sensibly altered when a load passes over the bridge; this being the case, it follows that the hangers exert an equal pull on the flexible member if they are at equal horizontal distances apart, otherwise the shape of this member would not remain parabolic. The stiffening girder is subjected to the action of the actual loads, to a uniform upward pull by the hangers (equal and opposite to what they exert on the cable), and to the reactions, upwards or downwards as the case may be, at the ends. If the stiffening girder is made strong enough to resist these forces, the flexible member can only alter its shape by the amount this girder deflects, except between the points of connection of the hangers. This shows why a stiffening girder which carries the load platform, and is suspended by hangers from the flexible parabolic member, provides a means of transmitting to the flexible member pulls which correspond to those due to a uniform load.

Flexible Cable Subject to Uniform Load

It has already been pointed out from the application of the methods of the equilibrium polygon, that the cable will hang in a parabolic curve, and that the depth at the centre $D = \frac{w\,l^2}{8\,H}$ or $H = \frac{w\,L^2}{8\,D}$, and any ray in the force diagram gives the total tensile stress at the point of the equilibrium polygon at which the tangent to it is parallel to the ray. The result may also be obtained from first principles, as follows :—w is the load per foot run which is uniformly distributed horizontally. Consider a length A B (Fig. 125) of the cable, where A is its centre point. Let x, y be the co-ordinates of B. The forces acting on A B are H horizontally at A, a tension T at B acting tangentially, and the uniform load $w\,x$ on A E $(= x)$, the centre of gravity of which will be at its centre C. Since the length of cable A B is in equilibrium under the action of the three forces H, T, and

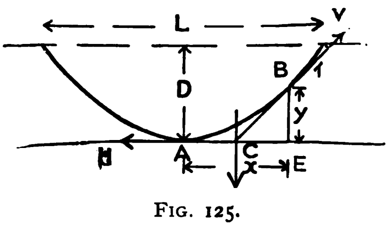

FIG. 125.

$w\,x$, the tangents at A and B must intersect on the vertical through C, and the Δ B E C is a triangle of forces for the three forces concerned; therefore

$$\frac{2\,y}{x} = \frac{w\,x}{\text{H}}, \quad \text{or } y = \frac{w\,x^2}{2\,\text{H}},$$

the equation to a parabola referred to its vertex.

At the point of support V, $x = \frac{\text{L}}{2}$ and $y = \text{D}$,

$$\therefore \qquad \text{D} = \frac{w}{2\,\text{H}} \times \frac{\text{L}^2}{4}, \quad \text{H} = \frac{w\,\text{L}^2}{8\,\text{D}}, \quad \text{and} \quad w = \frac{8\,\text{D}\,\text{H}}{\text{L}^2}.$$

Substituting for w, $\qquad y = \frac{4\,\text{D}}{\text{L}^2}\,x^2.$

Since H at A and the horizontal component of T at B are the only horizontal forces acting on A B, it follows that the horizontal component of the tension at any point $=$ H.

The tension at any point

$$= \sqrt{\text{H}^2 + w^2\,x^2} = w\sqrt{\frac{\text{L}^4}{64\,\text{D}^2} + x^2}.$$

The tension at the points of support

$$= \frac{w\,\text{L}}{2}\sqrt{\frac{\text{L}^2}{16\,\text{D}^2} + 1}, \text{ since } x = \frac{\text{L}}{2}.$$

Flexible Cable Provided with Stiffening Girder

Since the stiffening girder is suspended from the cable, it will be seen that it will be bent by any increase or decrease in the length of the latter. As the alteration in length due to change of temperature is very apparent, the stiffening girder, if continuous from end to end, would be very perceptibly deflected at the centre, and would thereby be stressed in proportion to the deflection. In order to see what stress might be induced at the centre of a stiffening girder due to a range of temperature of 60 deg. F. on either side of the mean, take the case of a span of 500 feet with a centre depth of 50 feet, the depth of the stiffening girder being 15 feet.

The length of half the cable

$$= s = \int_0^{\frac{\text{L}}{2}} \sqrt{d\,x^2 + d\,y^2} = \int_0^{\frac{\text{L}}{2}} \sqrt{1 + \left(\frac{d\,y}{d\,x}\right)^2}\,d\,x.$$

As just proved—

$$y = \frac{4\,D}{L^2}\,x^2; \therefore \frac{dy}{dx} = \frac{8\,D\,x}{L^2}.$$

$$\therefore \quad s = \int_0^{\frac{L}{2}} \left(1 + \frac{64\,D^2\,x^2}{L^4}\right)^{\frac{1}{2}} dx,$$

or $$s = \int_0^{\frac{L}{2}} \left(1 + \frac{64}{2}\,\frac{D^2\,x^2}{L^4} - \ldots\ldots\right) dx$$

$$= \frac{L}{2} + \frac{4}{3}\,\frac{D^2}{L}, \text{ neglecting the smaller terms.}$$

$$\therefore \quad \delta s = \frac{8}{3}\,\frac{D}{L}\,\delta D, \text{ or } \delta D = \frac{3\,L}{8\,D}\,\delta s.$$

It may be noticed in passing that since $H = \dfrac{w\,L^2}{8\cdot D}$,

$$\therefore \quad \delta H = -\frac{w\,L^2}{8\,D^2} \times \delta D = -H\,\frac{\delta D}{D},$$

from which the decrease of the stress due to a rise of temperature or the increase of the stress due to a fall of temperature can be ascertained.

For the particular case $s = 250 + \frac{4}{3} \times \frac{50}{10} = 256{\cdot}7$. Taking the coefficient of contraction $= a = 0{\cdot}000007$.

The increase of length due to a rise of 60 deg. F.

$$= 256{\cdot}7 \times 60 \times 0{\cdot}000007 = 0{\cdot}11;$$

$$\therefore \quad \delta D = \frac{3\,L}{8\,D} \times 0{\cdot}11 = \frac{3}{8} \times 1{\cdot}1 = 0{\cdot}4.$$

Now the deflection of a girder uniformly loaded (page 168) equals $\dfrac{5}{24}\,\dfrac{L^2}{E \times d} \times f_c$, where d is the depth of the stiffening girder and f_c is the flange stress at the centre.

$$\therefore \quad \frac{5}{24}\,\frac{L^2}{E \times d} \times f_c = 0{\cdot}4, \quad E = 13{,}000 \text{ tons per sq. in.}$$

$$f_c = \frac{0{\cdot}4 \times 24 \times 13{,}000 \times 15}{5 \times 500 \times 500} = 1{\cdot}5 \text{ tons per sq. in.}$$

And it will be seen that this is greater the greater the depth of the stiffening girder. It is obviously undesirable that stresses of such magnitude should be induced in the flanges of the stiffening girder without serving any useful purpose, but this effect may

be to a great extent eliminated by introducing a hinge, or its equivalent, at the centre of the stiffening girder, so that when the centre of the cable rises or falls, instead of the girder being deflected at the centre, the hinge allows the centre of the girder to accommodate itself to the new position of the cable without any bending taking place there. Hence the great advantage of the central hinge.

The cable is supported on saddles at A and B, carried on rollers working on a suitable bed-plate on the tops of the towers (Fig. 126), and is continuous over the saddles to the anchorages D, E,

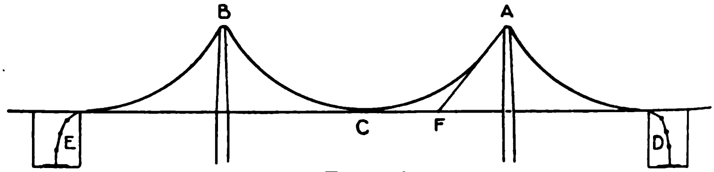

FIG. 126.

where the ends are secured to girders weighted down with concrete. The backstays may carry the roadway of the approach span just as the main cable carries the roadway of the main span.

If the angle of the backstay at the tower is the same as that of the cable on the main span side, the resultant tension in the cable will be the same on both sides of the saddle, and there will be no tendency for the cable to move relatively to the saddle. If the centre span only is loaded, a certain tension will be produced in the cable to the left of the saddle at A. This saddle will therefore move on the rollers towards B, as the load comes on, a distance sufficient to produce the same horizontal component of tension in the cable to the right of the saddle. If the angle of the backstay part of the cable at A be steeper than that of the main span cable, then the resultant tension in the cable to the right of A will be greater than the resultant tension in the cable to the left of A when the horizontal components of these tensions balance each other. There would therefore, under these circumstances, be a tendency for the cable to move relatively to the saddle unless the coefficient of friction is sufficient to prevent such motion. In this case the cable must be secured to the saddle in such a manner as to prevent any relative motion taking place. Suppose the angle of the backstay is the same

is that of the cable of the main span. If the centre span is loaded so as to produce an average intensity of stress in the cable of 12 tons per square inch, taking the same dimensions as in the above example, find how far the saddle A would move towards B before the horizontal components of stress in the cable on the two sides of A balance. Take the modulus of elasticity as 13,000 tons per square inch.

Now the tangent at A intersects the tangent at C (Fig. 126) in F, such that $\text{CF} = \frac{\text{L}}{4}$, therefore the horizontal movement of the saddle at A

$$= \frac{256 \cdot 7 \times 12}{13{,}000} \times \frac{\text{L}}{4\sqrt{\frac{\text{L}^2}{16} + \text{D}^2}} = 0 \cdot 2 \text{ foot.}$$

It will be observed that this movement of the saddles at A and B will cause the centre point C of the middle span to be lowered, which the hinge in the stiffening girders at this point admits of. But the shape of the flexible members between the saddles and C will also change, in some degree, notwithstanding the presence of that hinge. It is clear, therefore, that secondary stresses in the two halves of the stiffening girder, due to such alteration of shape, are to be expected.

Flexible Cable with Stiffening Girder Hinged at the Centre

In a suspension bridge, as in any other structure, the moment of resistance of the stresses about any point in a vertical section must be equal to the bending moment, which, in this case, is due to the horizontal components of the reactions at the points of support, as well as to the loads and vertical reactions. The ordinates to the centre line of the cable relative to the line joining the two supports multiplied by H equals the moment of the horizontal component of the reactions about the centre of the flexible member, and is therefore also the moment of the uniform pull of the hangers on the cable and of its vertical reaction, and the equilibrium polygon for the cable coincides with its centre line.

The stiffening girder (Fig. 127), in addition to being subjected to the upward pull, which is the reverse of the downward pull on the cable, is subjected to the load and the vertical reactions at its ends. The bending-moment diagram due to the first is,

of course, as before, the centre line of the cable, and if the bending-moment diagram for the loads and their reactions be plotted on the same base to the proper scale, the difference

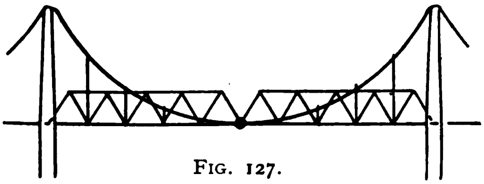

FIG. 127.

of the ordinates of the two diagrams will be the resultant bending moment on the stiffening girder. The scale is fixed from the fact that there can be no bending moment at the central hinge, therefore the moment at the centre of the loads and their reactions equals D × H, which gives the value of H and the scale of moments, and knowing H the force diagram gives the total tensile stress in the flexible member at any point due to the actual load.

The Effect of a Single Load on the Bridge

Let A O B (Fig. 128) be the flexible member, the ordinates of which multiplied by H—which has to be determined—equal the moments of the horizontal components of the reactions or the moments of the vertical pulls of the hangers and the vertical reaction. On A B plot the bending - moment diagram A E B for the single load Q, which will be a triangle with its apex on the vertical line through the position of Q, the scale being such that one side of this triangle passes through O, the centre of the parabola. Now O C = D and D × H = the bending moment of the load Q about the centre line O C. The

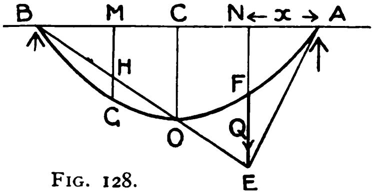

FIG. 128.

reaction at B due to Q $= \dfrac{Q \times x}{L}$;

∴ its bending moment at the centre

$$= \frac{Q \times x}{L} \times \frac{L}{2} = \frac{Q\,x}{2}.$$

$$\therefore\ D \times H = \frac{Q\,x}{2}, \quad \text{and } H = \frac{Q\,x}{2\,D},$$

which is the multiplier for the ordinates of the diagram (Fig. 128) to convert them into moments.

This value for H shows that any load Q increases the horizontal component of the tension in the cable by an amount $\frac{Q\,x}{2\,D}$, from which it is evident that the maximum tension occurs in the cable when the platform is fully loaded and, if the load is not uniform, when the heaviest part is situated at the centre.

For the position of the load in Fig. 128, it is clear that E F is the maximum positive bending moment due to it, and that G H at the centre of the other half of the girder is the maximum negative value.

Now E F = E N − F N. The moment E N $= D \times \frac{2\,(L - x)}{L} \times H$ $= \frac{Q\,x\,(L - x)}{L}$, and the moment F N $= \frac{w^1}{2}\,x\,(L - x)$, where w^1 = the pull of the hangers per foot run. Now $\frac{w^1\,L^2}{8} = \frac{Q\,x}{2}$, the moment at the centre;

$$\therefore\ w^1 = \frac{4\,Q\,x}{L^2}; \quad \therefore\ F\,N = \frac{2\,Q\,x^2\,(L - x)}{L^2}.$$

Thus the moment E F $= \frac{Q\,x\,(L - x)\,(L - 2\,x)}{L^2}$,—by differentiating this and equating to zero, it is found that its maximum value is when $x = \frac{L}{2}\,(1 \pm \sqrt{\tfrac{1}{3}})$.

Again G H = G M − H M. The moment G M $= \frac{w^1}{2} \times \frac{L}{4} \times \frac{3\,L}{4}$ $= \frac{3\,Q\,x}{8}$, and the moment H M $= \frac{1}{2}\,D \times H = \frac{Q\,x}{4}$.

∴ the moments G H $= \frac{Q\,x}{8}$. This is a maximum when x is a maximum—*i.e.* when $x = \frac{L}{2}$ when G H $= \frac{Q\,L}{16}$.

Thus the positions that a single load Q must occupy to give the maximum positive and the maximum negative bending moments have been found.

It is, however, of more importance to determine what dispositions of the load will give a maximum tension in the cable and maximum shearing forces and bending moments on the stiffening girder.

The first of these has been determined in a previous paragraph, where it is shown that to produce the maximum tension in the cable the load must cover the bridge with the heavier portions—if such exist—at the centre. If the bending moment at the centre be determined for such a disposition of the load and equated to $D \times H$, the value of H will be found, and the rays of the force diagram give the resultant tension at any point.

Maximum Shearing Force at any Section of the Stiffening Girder Hinged at the Centre

Let x in Fig. 129 be the distance from A of the section X at which the shearing force is required, and consider the effect of a single load Q distant ξ from A. R is the reaction at A due

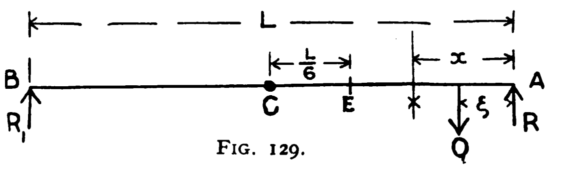

FIG. 129.

to the load Q and the vertically upward pulls of the hangers, and R^1 is the corresponding reaction at B. w^1 is the pull per linear foot of the hangers. Then for equilibrium the sum of the vertical forces = 0, and the moments of the acting forces about the hinge on either side = 0, since there is no resultant bending moment at the hinge—*i.e.*

$$R + w^1 L - Q + R_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R_1 \frac{L}{2} + \frac{w^1 L^2}{8} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$R \frac{L}{2} + \frac{w^1 L^2}{8} - Q\left(\frac{L}{2} - \xi\right) = 0 \quad . \quad . \quad (3)$$

Substituting for w^1 in (3) from (2) and dividing by $\frac{L}{2}$,

$$R - R_1 - Q\left(1 - \frac{2\xi}{L}\right) = 0 \quad . \quad . \quad . \quad (4)$$

bstituting for w^1 in (1) from (2)—

$$R - 4R_1 - Q + R_1 = 0, \text{ or } R - 3R_1 - Q = 0 \quad (5)$$

ultiplying (4) by 3 and subtracting (5), $R = Q\left(1 - \frac{3\xi}{L}\right)$.

Substituting for R in (5), $R_1 = -\frac{Q\xi}{L}$.

Substituting in (2), $w^1 = \frac{4Q\xi}{L^2}$ and $H = \frac{w^1 L^2}{8D} = \frac{Q\xi}{2D}$.

These three equations give the value of R and R_1 on the stiffening girder due to the load Q and to the upward pull of the hangers, and the value per foot run of this upward pull due to the load Q.

It is obvious that R becomes negative if $\xi > \frac{L}{3}$, also that the reaction at the end remote from the load is always negative.

If the load Q *is between* A *and* X.—The load Q is greater than the reaction at A.

$$\text{The upward force} = w^1 x = \frac{4Q\xi x}{L^2}.$$

$$\text{The downward force} = Q - Q\left(1 - \frac{3\xi}{L}\right) = \frac{3Q\xi}{L}.$$

$$\therefore \text{ shearing force at X} = \frac{Q\xi}{L}\left(\frac{4x}{L} - 3\right).$$

This is negative, as x is always $< \frac{3}{4}L$, because x is only measured to the centre. This position of load therefore causes a negative shear at X.

Load Q *between* X *and* C.—In this case the load does not come into the shear at X, and the reaction R is negative when

$$\xi > \frac{L}{3}.$$

$$\text{The upward force} = \frac{4Q\xi x}{L^2}.$$

$$\text{The reaction at A} = Q\left(1 - \frac{3\xi}{L}\right).$$

$$\therefore \text{ shearing force at X} = \frac{Q}{L^2}\{\xi(4x - 3L) + L^2\}.$$

This is positive if $\xi < \frac{L^2}{3L - 4x}$; if $x = 0$, the shear changes from positive to negative, when $\xi = \frac{L}{3}$; when $x = \frac{L}{4}$ and greater values, the shear remains positive till $\xi = \frac{L}{2}$—*i.e.* till the

length XC is completely covered; for values of x between 0 and $\frac{L}{4}$, the shear changes through zero from a positive to a negative value at a distance from A towards C

$$= \frac{L^2}{3L - 4x}.$$

When x is $> \frac{L}{4}$ the shear will be positive for loads from X to C.

Load Q *between* C *and* B.—The reaction at A is now downward and $= -\frac{Q\xi}{L}$ (ξ measured from B).

The upward pull of the suspension rods $= \frac{4Q\xi x}{L^2}$ (ξ measured from B).

$\therefore$ the shearing force $= \frac{Q\xi}{L^2}(4x - L)$, which is positive if $x > \frac{L}{4}$, negative if x is less $\frac{L}{4}$, and is zero if $x = \frac{L}{4}$, whatever the value of ξ may be—*i.e.* for any position of Q between C and B.

Thus, when $x > \frac{L}{4}$, the shearing force at X has its maximum positive value when the portion of the span to the left of X is covered, and its maximum negative value when the portion of the span to the right of X is covered.

When $x < \frac{L}{4}$, the shearing force at X has its maximum positive value when the length from X to a point distant $\frac{L^2}{3L - 4x}$ from A is loaded; and its maximum negative value when the length from this point to B and the length of A to X are loaded. For example, the maximum positive shear at the centre hinge occurs when the half span on the left is covered, and the maximum negative value when the half span on the right is covered.

The maximum positive shear at A occurs when the span is covered for one-third of its length from A, and the maximum negative shear at A when the span is covered for two-thirds of its length from B. The maximum positive shear at one-quarter the span is when the girder is loaded from that point to the centre, and the maximum negative shear when the girder is loaded from that point to the nearest support. It is immaterial whether the other half of the girder is loaded or not, as the

. of a load on that half of the girder, on the shear at the 'e of the first half, is nil.

'he values of the above shears will be worked out after ıdering the distribution of load that will give a maximum ıing moment at any section of the stiffening girder.

aximum Bending Moment at any Section of the Stiffening Girder

Let it be required to find the maximum bending moment : (Fig. 130) a distance x from A when a uniform load of w tons

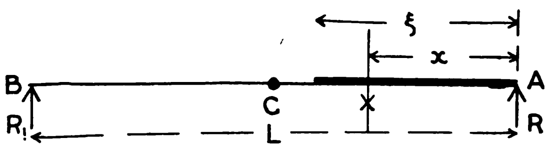

FIG. 130.

linear foot advances from A. Suppose the uniform load ers a distance ξ from A, where ξ is less than $\frac{L}{2}$, let w^1 be the formly distributed pull of the rods as before.

To obtain the equations of equilibrium, as before the alge-ic sum of all the forces resolved vertically = 0, and since bending moment at the central hinge is zero, the sum of moments of the forces on either side of the hinge about it ıals zero, using the same symbols :—

$$R + w^1 L - w \xi + R_1 = 0 \quad . \quad . \quad . \quad (1)$$

$$R_1 \frac{L}{2} + \frac{w^1 L^2}{8} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

l
$$R \frac{L}{2} + \frac{w^1 L^2}{8} - \frac{w \xi (L - \xi)}{2} = 0 \quad . \quad . \quad (3)$$

ɔstituting for w^1 in (3) from (2) and dividing by $\frac{L}{2}$,

$$R - R_1 - \frac{w \xi (L - \xi)}{L} = 0 \quad . \quad . \quad . \quad (4)$$

ɔstituting for w^1 in (1) from (2)—

$$R - 4 R_1 - w \xi + R_1 = 0, \text{ or } R - 3 R_1 - w \xi = 0 \quad . \quad (5)$$

ltiplying (4) by 3 and subtracting (5)—

$$R = \frac{w \xi}{2 L} (2 L - 3 \xi).$$

P

Substituting for R in (5)—

$$R_1 = -\frac{w\,\xi^2}{2\,L}.$$

Substituting in (2)—

$$w^1 = \frac{2\,w\,\xi^2}{L^2} \text{ and } H = \frac{w^1\,L^2}{8\,D} = \frac{w\,\xi^2}{4\,D}.$$

Now $$M_x = R \times x - (w - w^1)\frac{x^2}{2}$$

$$= \frac{w\,\xi\,x}{2\,L}(2\,L - 3\,\xi) - \frac{w\,x^2}{2}\left(1 - 2\frac{\xi^2}{L^2}\right)$$

$$= \frac{w}{2}\left\{\frac{\xi^2\,x}{L^2}(2\,x - 3\,L) + 2\,\xi\,x - x^2\right\}.$$

To find the value of ξ which makes this a maximum—*i.e.* to find how far the load must extend from A towards the centre to make the bending moment at X a maximum—we have—

$$\frac{d\,M_x}{d\,\xi} = 0, \text{ or } 2\,\xi\,x\,(2\,x - 3\,L) + 2\,L^2\,x = 0;$$

$$\therefore \quad \xi = \frac{L^2}{3\,L - 2\,x} \quad . \quad . \quad . \quad . \quad (6)$$

Substituting this value in the expression for M_x we have—

$$\text{Maximum } M_x = \frac{w}{2}\left\{-\frac{L^2\,x}{3\,L - 2\,x} + \frac{2\,L^2\,x}{3\,L - 2\,x} - x^2\right\}$$

$$= \frac{w}{2}\,\frac{L^2\,x - 3\,L\,x^2 + 2\,x^3}{3\,L - 2\,x},$$

$$\text{or maximum } M_x = \frac{w}{2}\,\frac{x\,(L - x)\,(L - 2\,x)}{3\,L - 2\,x} \quad . \quad . \quad . \quad (7)$$

This may be written—

$$= \frac{w\,\xi}{2} \times \frac{x\,(L - x)\,(L - 2\,x)}{L^2}.$$

If this be compared with the formula (page 221) for the bending moment at X due to a single load Q placed there, it will be seen that :—The maximum bending moment at X, due to a uniformly distributed load, equals that due to a concentrated load at X, whose weight is half the distributed load that gives the maximum bending moment there. The value of x that will cause it to have the greatest possible value can be found by differentiating equation (7) with respect to x.

From (6) $$\xi - x = \frac{L^2 - 3\,L\,x + 2\,x^2}{3\,L - 2\,x} = \frac{(L - x)\,(L - 2\,x)}{3\,L - 2\,x}.$$

Therefore the expression for maximum M_x, equation (7), may be written—

$$\text{Maximum } M_x = \frac{w}{2}\,x\,(\xi - x),$$

which is, of course, the expression for the bending moment at distance x from the end of a girder length ξ uniformly loaded. From equation (6) if $x = \frac{\text{L}}{4}\ \xi = \frac{2}{5}\text{L}$; $\therefore$ the maximum value of the bending moment at the centre of each stiffening girder, length $\frac{\text{L}}{2}$, equals $\frac{w}{2}\frac{\text{L}}{4} \times \frac{3}{20}\text{L} = \frac{3\,w\,\text{L}^2}{160}$.

The bending moment at X would be zero if the whole span were uniformly loaded, consequently the maximum negative bending moment at X will occur when the span is covered for a distance $= \text{L} - \xi$ from B, and its value will be the maximum negative bending moment at X $= -\frac{w}{2}\frac{x\,(\text{L}-x)\,(\text{L}-2\,x)}{3\,\text{L}-2\,x}$.

Value of Maximum Shearing Forces

The above expression for R, R_1, and w^1 when part of the span from one end of the bridge is uniformly loaded enables the maximum value of the shears referred to on page 224 to be calculated.

The maximum positive shear at the centre hinge occurs when the left half of the span is covered,

$$\therefore\ \text{R} = -\frac{w\,\text{L}}{8} \text{ and } w^1 = \frac{2\,w\,\text{L}^2}{4\,\text{L}^2} = \frac{w}{2};$$

$\therefore$ the maximum positive shear at the centre hinge

$$= -\frac{w\,\text{L}}{8} + \frac{w}{2}\frac{\text{L}}{2} = \frac{w\,\text{L}}{8}.$$

The maximum negative shear at the centre hinge occurs when the right half of the span is covered, and therefore

$$= \frac{w}{4}\frac{\text{L}}{\text{L}}\left(2\,\text{L} - \frac{3\,\text{L}}{2}\right) - \frac{w\,\text{L}}{2} + \frac{w}{2}\frac{\text{L}}{2} = \frac{w\,\text{L}}{8} - \frac{w\,\text{L}}{2} + \frac{w\,\text{L}}{4} = -\frac{w\,\text{L}}{8}.$$

Thus the maximum positive and negative shear at the centre hinge $= \pm\frac{w\,\text{L}}{8}$.

The shear at either support equals the reaction there, and therefore the maximum shear occurs when the reaction is a maximum.

Since $\text{R} = \frac{w\,\xi}{2\,\text{L}}(2\,\text{L} - 3\,\xi)$, it will be a maximum when $\frac{d\,\text{R}}{d\,\xi} = 0$ —*i.e.* when $2\,\text{L} - 6\,\xi = 0$, or when $\xi = \frac{\text{L}}{3}$; similarly when the rest of the span is loaded the maximum negative reaction at A occurs. This is the same result as arrived at before.

Then $R = \frac{w\,L}{6\,L}(2\,L - L) = \frac{w\,L}{6}$. Now when $x = \frac{L}{2}$, $R + R_1 = 0$—*i.e.* if the whole span is covered the reaction on the stiffening girder $= 0$—therefore the maximum positive and negative shears at the ends of the girder are $\pm \frac{w\,L}{6}$.

At one-quarter of the span the maximum positive shear occurs when the load extends from that point to the centre, and the maximum negative shear when loaded from that point to the near end. Taking the latter first, $R = \frac{w\,L}{8\,L}\left(2\,L - \frac{3\,L}{4}\right)$, or $R = \frac{5\,w\,L}{32}$ and $w^1 = \frac{2\,w\,L^2}{16\,L^2} = \frac{w}{8}$, therefore the maximum negative shear $= \frac{5}{32}\,w\,L + \frac{w}{8}\,\frac{L}{4} - \frac{w\,L}{4} = -\frac{w\,L}{16}$.

Now if the half span is covered the shear at $\frac{L}{4} = 0$; $\therefore$ the maximum positive and negative values of the shear at

$$\frac{L}{4} = \pm \frac{w\,L}{16}.$$

HINGED-RIB SUSPENSION BRIDGE

Instead of using a flexible member and a stiffening girder as two separate entireties, they may be combined into a single member, and the bridge will then consist of two ribs (Fig. 131), hinged at the supports and at their junction at the centre, which correspond to both the flexible member and the stiffening girder. This makes any deflection, except that due to strains, impossible, and therefore produces a more rigid structure. The dead load, if uniformly distributed, will cause a tension in the flanges of the rib corresponding to that produced in the flexible member in the previous case; but for the live load, since tensions are produced in the flanges of the ribs corresponding to the tensions in the flexible member, whilst at the same time they are subjected to

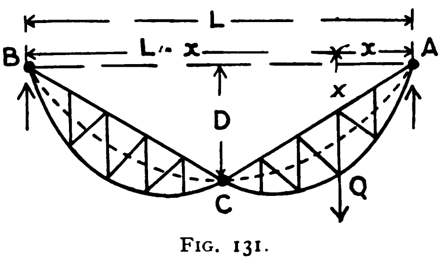

FIG. 131.

the bending moment (in the former case taken up by the stiffening member), it follows that the stresses in the flanges will not be a maximum when the bending moment (which, of course, would be the same in this case) is a maximum. Hence a different method of treatment is desirable. The determination of the maximum stresses in the members in this case is exactly the same problem as that of the three-hinged arch.

There is, however, one type which can be more easily dealt with in a different manner, which remark, of course, also applies to the corresponding arch; and that is, when the centre line of the ribs is a parabola and the upper flanges are straight lines joining the centre hinge to the two supporting hinges. The lower flanges are of course curved, so that at any vertical section the lower flange is the same distance below the centre line as the straight upper flange is above it.

Dead Load Stresses.—Suppose the dead load is equivalent to w tons per foot run, the total load on each rib is then $\frac{w\,\mathrm{L}}{2}$.

We may imagine the structure to be made up of two independent structures. (1) The two ribs, each carrying a uniformly distributed load equal to $\frac{w\,\mathrm{L}}{2}$ with vertical reactions at their two ends, each equal to $\frac{w\,\mathrm{L}}{4}$; and (2) the triangular frame A C B carrying a load $\frac{w\,\mathrm{L}}{2}$ at C equal to the two reactions $\frac{\mathrm{W\,L}}{4}$ of the two ribs at C. It is clear that the curved flanges of the ribs and the web members of the ribs only form part of (1), and the stresses in them may therefore be found in exactly the same manner as for a parabolic girder supported at the two ends. The supports are not level, but the horizontal component of stress in the flanges will be $\frac{w\,l^2}{8\,d}$ as before, where l is the horizontal length of the rib and d its central vertical depth—*i.e.* $l = \frac{\mathrm{L}}{2}$ and $d = \frac{\mathrm{D}}{2}$, because the depth to the parabolic centre line from the middle points of A C or B C $= \frac{\mathrm{D}}{4}$.

The straight flanges of the two ribs A C and B C are obviously a component part of both (1) and (2), and the stresses in them are therefore the resultant of the stresses due to the load on

each structure considered separately. These members will be in compression when considered as a part of the ribs, and the amount of the horizontal component of this compression would be $\dfrac{w\left(\frac{L}{2}\right)^2}{8\frac{D}{2}} = \dfrac{w\,L^2}{16}$. This would, of course, also be the horizontal component of the tension in the curved flanges of the ribs.

As a part of the structure A C B with the load $\dfrac{w\,L}{2}$ hanging at C, A C and B C will be in tension, the horizontal components of the tension being $\dfrac{w\,L}{4} \times \dfrac{L}{2\,D} = \dfrac{w\,L^2}{8\,D}$. The resultant stress in A C and B C will therefore be a tension $= \dfrac{w\,L^2}{8\,D} - \dfrac{w\,L^2}{16\,D} = \dfrac{w\,L^2}{16\,D}$. Thus there would be a resultant horizontal tension in both the straight and curved flanges of the ribs $= \dfrac{w\,L^2}{16\,D}$.

Live Load Stresses.—In the case of the curved flanges and the web members, the maximum live load stresses will, as before, be exactly the same as for a parabolic girder of horizontal length $\dfrac{L}{2}$ and vertical depth $\dfrac{D}{2}$, and may be found by scaling the elevation as explained in Chapter IV. In order to find the maximum compression and tension in A C and B C due to the live load, it is best to consider the effect of a single load Q at each panel point consecutively on either of the ribs. If a load Q (Fig. 131) acts at a distance x from A, it will cause a reaction R at A $= \dfrac{Q\,(L - x)}{L}$ and a reaction R_1 at B $= \dfrac{Q\,x}{L}$. The expression for R may be written $Q\left(1 - \dfrac{2\,x}{L}\right) + \dfrac{Q\,x}{L}$, the first term being the reaction due to a load Q on the rib A C at a distance x from A, and the second term is the same as the reaction R_1 at B.

Thus the load Q on the structure span A B may be considered as a load Q on the rib A C with reactions at A and C equal to $Q\left(1 - \dfrac{2\,x}{L}\right)$ and $Q\,\dfrac{2\,x}{L}$ respectively, and a load $\dfrac{2\,Q\,x}{L}$ on the structure A C B acting at C equal to the reaction of the rib A C at C.

The load Q at X on the rib A C will cause a compression in A C at X, whose horizontal component $= \dfrac{Q\left(1 - \dfrac{2x}{L}\right)x}{d^1}$, where d^1 is the depth of the rib at the panel point considered. For any other panel point to the right of X distant ξ from A, the horizontal component of stress in the flange $= \dfrac{Q\left(1 - \dfrac{2x}{L}\right)\xi}{d^1}$; and for any panel point to the left of X distant ξ^1 from C the horizontal component of flange stress $= \dfrac{2\,Q\,x\,\xi^1}{L\,d^1}$, d^1 being the depth of the rib at the panel point considered. The load $\dfrac{2\,Q\,x}{L}$ on the structure A C B will cause a tension in A C at X, whose horizontal component $= \dfrac{Q\,x}{L} \times \dfrac{L}{2\,D} = \dfrac{Q\,x}{2\,D}$, and this will be the same with the load at X for all the panel points.

If the above values of the horizontal component of the compression due to A C forming part of the rib be worked out for each panel point with the load at X, and the above horizontal component of tension $\dfrac{Q\,x}{2\,D}$, due to A C being also a portion of the triangular structure A C B, be subtracted in each case, the result will be a compression or tension for each panel length. Let this be repeated for the load at each of the panel points of the rib A C, and let all the compressions and all the tensions be added separately for each bay, the result will be the maximum horizontal component of compression in the upper flange for each bay of the rib. The sum of the tensions at each point would obviously be increased if the rib B C be loaded, as this would increase the load at C for the triangular structure A C B by the amount $\Sigma \dfrac{2\,Q\,x}{L}$, where the Σ denotes the summation of $\dfrac{2\,Q\,x}{L}$ for each panel point; this load would cause a tension in C A equal to $\Sigma \dfrac{Q\,x}{2\,D}$, which must therefore be added to the sum of the tensions found at each panel point of A C in order to obtain the maximum tensions in each bay length of the upper flange of that rib.

CHAPTER XII

EARTH PRESSURES AND RETAINING WALLS

IN determining the amount of the earth pressure that may be expected against the back of a retaining wall, the first consideration is the value of the coefficient of friction in the interior of the mass of earth. This coefficient is not always the same thing as the tangent of the angle of the natural slope of the material; besides, the latter is not always readily determined, as some material takes a long time before it assumes its true angle of slope. A more satisfactory value of this coefficient of friction may be obtained in any particular case by filling a box without top or bottom with the earth and placing it upon a levelled surface of similar earth, and then finding the pull required to start the box and the contained earth into motion. A horizontal cord passing over a vertical pulley, with a scale pan at the other end to hold the necessary weights, provides a ready method of doing this. The earth may be loaded by placing weights upon it in the open box in order that the variation of the friction at various pressures may be observed. Of course care must be taken to avoid any friction on the edges of the box itself. The coefficient would equal the weight of the scale pan plus the weight in it divided by the weight of earth, etc., in the box. Having found the coefficient of friction for the earth in question, the method in Chapter VI. may be used to find the relation between the two principal stresses in the earth—*i.e.* between the maximum vertical and the horizontal lateral pressure of the earth. The plane which would be nearest the limit of stability would be that on which the resultant stress makes the greatest angle with the normal to the plane, and it is obvious that this angle must not be greater than the angle of friction, otherwise the mass of earth will be unstable. It was shown on page 123, that denoting this angle by $\theta - \psi$—

$$\text{Sin}\,(\theta - \psi) = \frac{p_x - p_y}{p_x + p_y}.$$

Now if we call ϕ the angle of friction it is clear that $\theta - \psi$ must be less than ϕ or in the limit equal to it. Therefore for equilibrium $\dfrac{p_x - p_y}{p_x + p_y} =$ or $< \sin\phi$.

Forming a new fraction by adding and subtracting the numerator and denominator on either side of the equation, we get—

$$\frac{p_x}{p_y} = \text{or} < \frac{1 + \sin\phi}{1 - \sin\phi}.$$

$$\therefore \qquad p_y = > p_x \frac{1 - \sin\phi}{1 + \sin\phi}.$$

Thus for equilibrium p_y must at least equal $p_x \times \dfrac{1 - \sin\phi}{1 + \sin\phi}$, that is to say, a wall to resist the earth pressure must be capable of withstanding a pressure of this amount. If w is the weight of 1 cubic foot of the earth and h is the depth below its horizontal surface to the point where the pressure is under consideration, the intensity of vertical pressure at that depth $=$ $w\,h$, and therefore from the above the horizontal lateral pressure equals $\dfrac{1 - \sin\phi}{1 + \sin\phi} \times w\,h$. Since h is the only variable, the total horizontal pressure per foot length that the earth would exert on a wall of height H to the top of the earth $= \dfrac{1 - \sin\phi}{1 + \sin\phi} \times \dfrac{w\,\text{H}^2}{2}$, and it will obviously act at one-third the height of the wall from the base, as in the case of water pressure. This is the value given by Rankine when the earth has a surface level with the top of the wall, but it will be observed that no notice has been taken of the friction on the back of the wall in arriving at the ratio of the lateral to the vertical pressure, therefore the lateral pressures given by this formula are too high.

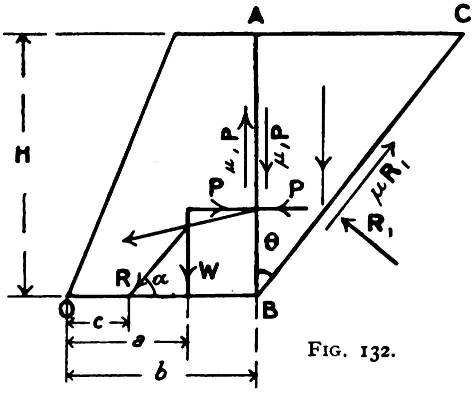

FIG. 132.

An entirely distinct method of determining the maximum

possible amount of the earth pressure on a wall which enables the effect of this friction to be taken into account is to imagine the earth to crack along a sloping line B C in Fig. 132, making an angle θ with the vertical, and to determine what inclination of this crack would cause a maximum pressure to be exerted on the wall. It is clear that the pressure against the wall would be greater if the earth cracks along a sloping line than if it cracks in a zigzag manner, as every bench in the line of the crack would form a partial support and reduce the pressure against the wall. The horizontal pressure P and the vertically upward friction μ_1 P exerted by the wall on the wedge of earth are written outside that wedge in Fig. 132, and the equal and opposite forces exerted by the earth on the wall are similarly written outside the wall. If we find an expression for P in terms of θ, and then determine the value of θ, which makes this expression a maximum, we shall obtain the maximum value that the pressure on the wall can possibly attain—for pipes are built into, or openings left through, the base of the wall and surrounded by gravel at the back to allow any water a free egress, so that it is impossible for it to back up against the wall and subject it to water pressure. Now as the wedge of earth by its motion tends to overturn the wall about the outer toe O, the friction between the earth and the back of the wall would tend to resist that overturning—*i.e.* it acts downward on the wall and upward on the wedge of earth. The pressure exerted by the wedge of earth A B C against the wall will be the resultant of its weight, the friction along B C and B A and the reaction R_1 normal to B C, and is, of course, equal and opposite to the resultant P of the supporting forces exerted by the wall on this wedge. Call μ the coefficient of friction of the earth on earth, μ_1 that of the earth on the wall. The weight of the wedge of earth

$$= \frac{H \times H \tan\theta}{2} \times w = \frac{w H^2 \tan\theta}{2}.$$

Resolving horizontally $P + \mu R_1 \sin\theta = R_1 \cos\theta$, and resolving vertically

$$\mu_1 P + \mu R_1 \cos\theta + R_1 \sin\theta = \frac{w H^2 \tan\theta}{2},$$

from which $P = \dfrac{1 - \mu\tan\theta}{1 - \mu\mu_1 + (\mu + \mu_1)\cot\theta} \times \dfrac{w H^2}{2}.$

Since some of the earth generally clings to the wall we may take $\mu_1 = \mu$, then

$$P = \frac{1 - \mu\tan\theta}{1 - \mu^2 + 2\mu\cot\theta} \times \frac{w H^2}{2} \quad . \quad . \quad . \quad (1)$$

In order to obtain the value of θ which causes this expression to be a maximum, differentiate with respect to θ and equate to zero, when it is found that for the maximum value—

$$\tan\theta = \frac{-2\mu + \sqrt{2(1+\mu^2)}}{1-\mu^2} \quad . \quad . \quad (2)$$

This therefore gives the angle θ of the crack which would produce a maximum horizontal pressure against the back of the wall, and substituting from (2) in (1) we get—

$$P = \frac{1 + 3\mu^2 - 2\mu\sqrt{2(1+\mu^2)}}{(1-\mu^2)^2} \times \frac{w\,H^2}{2},$$

and it acts at the height $\frac{H}{3}$ because the total pressure is equal to that produced by water multiplied by a constant.

The above investigation applies to the pressure of earth when its surface is level with the top of the wall, but if the earth slopes upwards from the wall at an angle ψ (Fig. 133), the wall is said to be a "surcharged" retaining wall. The resultant pressure may be obtained in an analagous manner to the last case. The equations of equilibrium will be the same as before, except that the expression for the weight of the wedge of earth

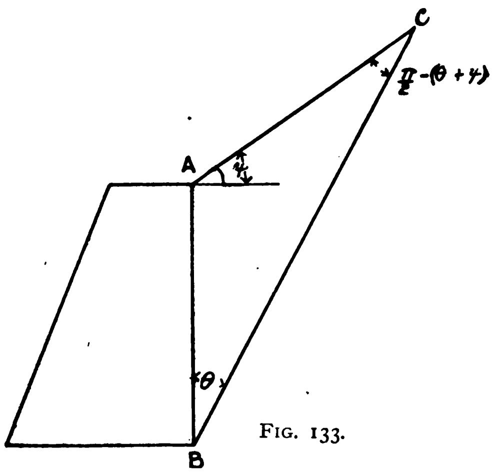

FIG. 133.

will now be $w \times \frac{H \times \overline{AC}\cos\psi}{2}$ and

$$\overline{AC} = H \times \frac{\sin\theta}{\cos(\theta+\psi)};$$

therefore the weight of the wedge of earth

$$= \frac{w\,H^2}{2} \times \frac{\sin\theta\cos\psi}{\cos(\theta+\psi)} = \frac{w\,H^2}{2} \times \frac{\tan\theta}{1-\tan\theta\tan\psi}$$

—*i.e.* it equals its previous value multiplied by $\frac{1}{1-\tan\theta\tan\psi}$,

therefore the expression for P is the value in equation (1) multiplied by this factor—*i.e.*

$$P = \frac{1}{1 - \tan\theta \tan\psi} \times \frac{1 - \mu\tan\theta}{1 - \mu^2 + 2\mu\cot\theta} \times \frac{w H^2}{2} \quad . \quad (3)$$

In order to find the value of θ which causes P to become a maximum, we must differentiate (3) with respect to θ and equate to zero as before, and instead of the value for $\tan\theta$ in equation (2) we now find that for a maximum value

$$\tan\theta = \frac{2\mu^2 - \sqrt{2\mu(1+\mu^2)(\mu - \tan\psi)}}{(1+\mu^2)\tan\psi - \mu(1-\mu^2)}.$$

It will be noticed that, if $\psi = 0$, this expression reduces to that in equation (2). The expression for P may be written

$$P = K\frac{w H^2}{2},$$

where the coefficient K has the above values and depends simply on the coefficient of friction and the angle ψ, and equals the ratio of the lateral to the vertical pressure, taking into account the friction on the back of the wall. The value of K for a level surface of earth and for earth slopes of 3 horizontally and 2 horizontally to 1 vertically are given in the following table, also the values of the ratio $\frac{1-\sin\phi}{1+\sin\phi}$ previously deduced:—

Material.	Angle of Friction ϕ.	Tan $\phi = \mu$.	K.			$\frac{1-\sin\phi}{1+\sin\phi}$.
			$\psi = \tan^{-1}\frac{1}{2}$.	$\psi = \tan^{-1}\frac{1}{3}$.	$\psi = 0$.	
Damp sand	22°	0·4	—	0·58	0·37	0·45
Earth	31°	0·6	0·44	0·33	0·25	0·32
Shingle or dry sand	39°	0·8	0·26	0·22	0·17	0·23
Rubble or well-drained clay	45°	1·0	0·17	0·15	0·125	0·17

It is evident from this table that the values of the coefficient given by Rankine's method and tabulated in the last column are higher than those given by the latter method, as might be expected—in fact, it will be seen that the values given by the former method are sufficient to allow for a certain amount of surcharge.

RETAINING WALL

To apply this to the case of a retaining wall the conditions of stability are (1) that the base shall be prevented from sliding; (2) that the moment of the overturning couple formed by the thrust of the earth behind it, multiplied by the distance of its centre of pressure above the base of the wall, shall be less than the moments of the equilibrating couples, to an extent which will allow the resultant force acting on the base to fall inside the middle one-third of its width. The equilibrating couples are,—that of the weight of the wall into the distance of its centre of gravity from the outer toe and that of the frictional resistance to vertical motion developed by the pressure of the earth at the back of the wall into the breadth of the base. For if N is the normal component of the resultant, and A the area of the base of unit length, $\frac{N}{A}$ is the average intensity of normal stress on the base, and if the resultant is applied at a distance c from the outer toe—*i.e.* at a distance $\frac{b}{2} - c$ from the centre—the tension due to the moment

$$N \times \left(\frac{b}{2} - c\right) = \frac{N \times \left(\frac{b}{2} - c\right)}{I} \times \frac{b}{2},$$

where I is the moment of inertia about an axis through the centre normal to the cross-section. Therefore for no tension c must not be less than the value given by the equation

$$\frac{N}{A} = \frac{N}{I}\left(\frac{b}{2} - c\right)\frac{b}{2},$$

or $$\frac{2\,I}{A\,b} = \frac{b}{2} - c, \text{ or } c = \frac{b}{2}\left(1 - \frac{4\,k^2}{b^2}\right) \quad . \quad . \quad . \quad (1)$$

where k is the radius of gyration about the same axis as I. Thus for a rectangular base $k^2 = \frac{b^2}{12}$, $\therefore c = \frac{b}{3}$.

Note.—The same formula (1) would apply to any structure symmetrical about two axes—*e.g.* for a circular solid column subject to wind pressure when $k^2 = \frac{b^2}{16}$, $\therefore c = \frac{b}{2} \times \frac{3}{4} = \frac{3}{8}b$—*i.e.* the resultant must fall in the middle one-quarter.

Call W the weight of the wall of unit length, let a be the distance of the vertical through its centre of gravity from O (Fig. 132), and let c be the distance from O to the point where $-$ R the reaction of the foundation on the wall (equal and opposite

to the resultant of the remaining forces acting on it) cuts the base. Thus R is the resultant of the weight of the wall and of the resultant of P and μ P acting on the back of the wall. Call a the inclination of R to the base, then taking moments about O we have—

$$\text{R} c \sin a = \text{W} a + \mu \text{P} b - \text{P}\frac{\text{H}}{3} \quad . \quad . \quad . \quad (2)$$

Resolving horizontally $\text{R} \cos a = \text{P}$

and resolving vertically $\text{R} \sin a = \text{W} + \mu \text{P}$.

$$\therefore \qquad \tan a = \frac{\text{W}}{\text{P}} + \mu.$$

Substituting for R sin a in (2)

$$c(\text{W} + \mu \text{P}) = \text{W} a + \mu \text{P} b - \text{P}\frac{\text{H}}{3}$$

$$c = \frac{\text{W} a + \mu \text{P} b - \text{P}\frac{\text{H}}{3}}{\text{W} + \mu \text{P}} \qquad (3)$$

In order to avoid tension at the inner toe c must be equal to or greater than $\frac{b}{3}$, as already pointed out. The top width of the wall must be chosen so as to be consistent with durability in the situation under consideration. The shearing strength of the wall at any horizontal section must be ample to resist the horizontal thrust above its plane; this question of shearing strength will be considered more fully in connection with dams. As an example of the method of utilising the above method, take the case of a retaining wall 20 feet high with vertical back and sloping face, the top width being 4 feet and the width at the base 7 feet; if the sine of the angle of friction of the earth is 0·65 and the tan 0·85, and its weight is 110 lb. per cubic foot, and that of the masonry of the wall 150 lb. per cubic foot, find whether the wall satisfies the condition of there being no tension at the inner toe—using Rankine's formula for earth pressure. The total pressure

$$= \text{P} = \frac{1 - \sin \phi}{1 + \sin \phi} \times \frac{w \text{H}^2}{2}$$

$$= \frac{0{\cdot}35}{1{\cdot}65} \times \frac{110 \times 400}{2} = 4{,}700 \text{ lb.}$$

The weight of the wall $= \text{W} = \frac{11}{2} \times 20 \times 150 = 16{,}500$ lb.

The distance of its C G from O

$$= a = \frac{\frac{20 \times 3}{2} \times \frac{2}{3} \times 3 + 4 \times 20 \times 5}{11 \times 10} = \frac{460}{110} = 4{\cdot}2 \text{ feet.}$$

From (3) $c = \dfrac{16{,}500 \times 4{\cdot}2 + 0{\cdot}85 \times 4{,}700 \times 7 - 4{,}700 \times \frac{20}{3}}{16{,}500 + 0{\cdot}85 \times 4{,}700}$

$= 3{\cdot}2$ feet.

Now one-third the width of the base $= \frac{7}{3} = 2{\cdot}3$ feet, therefore the resultant R falls well inside the middle third, and the base is unnecessarily wide.

If P were calculated by the second method given it would be found that the total earth pressure is less than that taken above, consequently the resultant pressure would fall still further inside the middle third of the base. Since the point at one-third the width of the base from the outer toe is the limiting position for the resultant to approach that toe in order that the condition of no tension at the inner toe may be

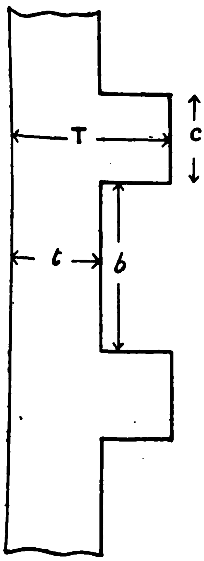

FIG. 134.

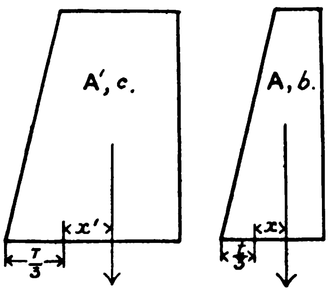

FIG. 135.

satisfied, this point may be called the centre of resistance, and moments may be taken about it instead of about the outer toe. The resulting moment about this point must either be zero or of the same sign as the equilibrating moments—*i.e.* the moments of the weight of the wall and of the friction at the back of the wall—the sum of which moments is called the moment of stability; or, in other words, these moments must be greater

than the overturning moment about that point — *i.e.* the moment of the earth pressure.

RETAINING WALL WITH COUNTERFORTS

Instead of a retaining wall being of constant cross-section, it is sometimes built with counterforts—*i.e.* of wide piers, with thinner portions between them, the latter being sometimes curved convex to the earth so that the stresses may be entirely compressions. An approximate method usually employed to find whether a proposed design satisfies the conditions of stability is (1) to find the equivalent wall of uniform cross-section, and (2) to ascertain whether the latter is stable.

Let Fig. 134 represent a horizontal section of such a wall, and let its vertical section be rectangular.

If P is the total earth pressure per unit length of the wall and μ the coefficient of friction of the earth on the back of the wall, if the friction on the sides of the counterforts be neglected, the moment of stability of the narrow part

$$= w\,\mathrm{H}\,b\,t \times \frac{t}{6} + \mu\,\mathrm{P}\,b \times \tfrac{2}{3}\,t,$$

and that of the wider part

$$= w\,\mathrm{H}\,c\,\mathrm{T} \times \frac{\mathrm{T}}{6} + \mu\,\mathrm{P}\,c \times \tfrac{2}{3}\,\mathrm{T}.$$

Therefore the moment of stability per unit length

$$= \frac{w\,\mathrm{H}}{6}\,\frac{b\,t^2 + c\,\mathrm{T}^2}{b + c} + \tfrac{2}{3}\,\mu\,\mathrm{P}\,\frac{b\,t + c\,\mathrm{T}}{b + c} \quad . \quad . \quad (1)$$

Let t_1 be the thickness of the equivalent wall of uniform cross-section having the same moment of stability, then the moment of stability of the equivalent wall per unit length

$$= \frac{w\,\mathrm{H}}{6}\,t_1^{\,2} + \tfrac{2}{3}\,\mu\,\mathrm{P}\,t_1 \quad . \quad . \quad . \quad (2)$$

By equating (1) and (2) the thickness of the equivalent wall of uniform thickness is determined, and its stability can be investigated by the same methods as before.

If the section of the wall is trapezoidal (Fig. 135), let A be the cross-sectional area of the narrow part, t the width of its base and x the distance of the vertical line through its centre of gravity from the point distant $\frac{t}{3}$ from the outer toe, and b its length; let A^1 be the cross-sectional area of the wider part, T the width of its base and x^1 the distance of the vertical line

ırough its centre of gravity from the point distant $\frac{T}{3}$ from ıe outer toe, and c its length, the portion of the moment of tability due to its weight would then be

$$\frac{w\,\mathrm{A}\,b\,x + w\,\mathrm{A}^1\,c\,x^1}{b+c}$$

ınstead of the first term in (1).

Utilisation of the Weight of the Earth to Increase Stability

In some designs the weight of part of the earth is utilised :o increase the stability of the wall; in one way, by building nto the wall horizontal shelves which project at the back, the weight of earth on the projecting shelf obviously increases the moment of stability, also the equilibrating frictional moment s increased, because the friction now acts on the vertical plane through the back of the shelves, and in this manner a wall of thinner cross-section may be rendered stable without any tension being introduced.

Another method of constructing a retaining wall which enables the weight of the earth to be utilised, is to construct the wall of a vertical and a horizontal slab of concrete, conıected together at the angle and reinforced with steel rods ıear the face that would come into tension. Narrow vertical ribs or counterforts are introduced about every two yards, filling the space between the vertical and horizontal slabs. Vertical and horizontal angles are built into the latter respectively at these sections, to which the reinforcing steel rods in the vertical and horizontal slabs are connected, and to the other flanges of the angles are fastened rods, which are embedded in the concrete of the ribs which anchor the vertical slab to the horizontal one. The weight of the earth over the horizontal slab holds the latter down, and enables the vertical slab to be tied back to it at the ribs.

CHAPTER XIII

FOUNDATIONS AND FOOTINGS

Two main considerations have to be taken into account when designing the foundations for a pier, abutment, retaining wall, or any similar structure :—(1) The foundation must be able to support the structure without appreciable compression, the base of the structure being widened out by footings so as to reduce the intensity of pressure on the foundation to allow of this. The pressure on the foundation may vary from 10 or more tons per square foot in the case of rock to 1 ton or less per square foot in the case of yielding material; but if the material on which the structure is to be built is not capable of carrying the intensity of pressure to which it must be subjected by the weight of the structure and its load without dangerous compression, the foundation would have to be reinforced either by piles, wells, cylinders, or caissons, as may be most convenient in any particular case, driven or sunk into the strata below. If there is a hard stratum at an accessible depth, the piles, cylinders, etc., would be carried down to that stratum. In such a case wells, cylinders, or caissons would be sunk as far as possible by grabbing the material from the inside, if the strata passed through permits of this method of proceeding. It is often necessary to resort to the help of divers if boulders or other such obstacles are met with under the cutting edge, and in some cases an air-tight deck has to be fixed and the compressed air method adopted, particularly for obtaining a good foundation for the cylinders, etc. In the case of such a foundation, not only is additional supporting force obtained from the surface friction of the piles or cylinders, but also from the solid stratum on which they rest. In case there is no solid stratum at a depth which is accessible under the circumstances, the surface friction of the piles, wells, or cylinders has to be depended upon, and in such case the sinking can generally be effected by grabbing the material from inside. The surface friction generally amounts from 2 to 10 cwt. per square foot of

surface according to the depth to which they are sunk and the material passed through.

(2) In addition to the supporting force the lateral stability of the foundations has to be considered if they consist of sand, gravel, clay, or such-like material. A formula has already been obtained (page 233) for the ratio of the principal stresses in earth under pressure—*i.e.* for the ratio of the minimum value of the smaller principal stress to that of the greater principal stresses in order that the material may be stable. In a foundation the maximum stress is often the vertical pressure on the foundation, and the formula gives the necessary value for stability of the stress at right angles to this—*i.e.* in the horizontal direction. This horizontal stress has again to be kept in equilibrium by the smaller stress at right angles to it; the latter stress would therefore be in the vertical direction, and is due to the weight of earth equal to the depth of foundation. Therefore the pressure of the earth due to the depth of the foundation must be great enough to bear the same ratio to the lateral pressure at the foundation that the latter does to the vertical pressure on the foundation, and this ratio must be at least equal to that given by the formula for the particular material. In the case of wet sand the lateral stability can be greatly increased by driving sheet piling all round the foundation to confine it.

It has been shown in Chapter XII. that for stability the ratio of the minimum to maximum stress at any point of the material composing the foundation must be equal to or greater than $\frac{1 - \sin \phi}{1 + \sin \phi}$—*i.e.* that the lateral pressure p_1 at right angles to the pressure on the foundations must for equilibrium be not less than this ratio of the intensity of compression p, at the edge of the foundation. Now if d is the depth of the foundation level below the ground line the pressure due to this depth of the earth equals $w\,d$, and this must not be less than the above fraction of the lateral pressure. We have therefore as the minimum values,

$$p_1 = \frac{1 - \sin \phi}{1 + \sin \phi} \times p,$$

and

$$w\,d = \frac{1 - \sin \phi}{1 + \sin \phi} \times p_1,$$

∴ the limiting value of the depth is given by the expression

$$w\,d = \left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)^2 \times p,$$

which is generally most conveniently applied in the form

$$\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2 \times d = \frac{p}{w} \quad . \quad . \quad . \quad (1)$$

As an example of the use of this formula, take the case of a pier 6 feet wide, carried on a base of concrete 8 feet wide, the intensity of pressure on the concrete due to the wall being 4 tons per square foot; the weight of 1 cubic foot of the concrete is 125 lb., and the weight of 1 cubic foot of the earth 106 lb.; the co-efficient of friction of the earth is 0·8. Find the depth to which the concrete should be carried for lateral stability of the foundation.

$$\operatorname{Tan}\phi = \mu = 0{\cdot}8; \quad \therefore \sin\phi = 0{\cdot}63.$$

The intensity of pressure at the base of the concrete

$$= p = \frac{6 \times 4 \times 2240 + 8 \times d \times 125}{8} = 6720 + 125\,d.$$

Now $\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2 = \left(\frac{1{\cdot}63}{0{\cdot}37}\right)^2 = (4{\cdot}4)^2 = 19{\cdot}4,$

$\therefore$ from (1) $\quad 19{\cdot}4 \times d = \frac{6720 + 125\,d}{106}$

$$= 63{\cdot}4 + 1{\cdot}2\,d.$$

$\therefore \quad 18{\cdot}2\,d = 63{\cdot}4,$

or $\quad d = 3{\cdot}5$ feet.

Since the angle of friction is not generally very accurately known for the material, and since it is not absolutely constant even at different parts of the material, the value of the ratio of the maximum intensity of stress to the minimum intensity must be recognised as an approximation only. When this is squared any error would be more than doubled proportionately, so that in calculating the necessary depth of a foundation in order to be stable against lateral movements, the result must only be regarded as an approximate value.

In the case of a retaining wall the maximum principal stress is not the compressive stress normal to the base at the edge, but is greater than this, because there is also a shearing stress on the base whose intensity is equal to the pressure of the earth on unit length of the wall divided by the width of the wall. From Chapter VI., page 116, the maximum principal stress would be $\frac{p + \sqrt{p^2 + 4q^2}}{2}$, if p is the normal intensity of compression and q is the shearing stress; this, however, does not make a very appreciable increase in the value of the principal

stress, and it is generally sufficiently accurate to take the normal stress.

As a further example, find to what depth the foundation of a retaining wall, 20 feet high above the ground line, should be carried for lateral stability of the foundation; the weight of a cubic foot of the earth is 110 lb. and the ratio of the principal stresses 4·7, and it is found that the maximum compression at the outer toe is 4,710 lb. per square foot. As the foundation is carried down the resultant pressure on the back of the wall will increase as the square of the depth, and the base must be widened at the front below the ground level to keep the point of intersection of the resultant force at one-third its width. The weight of the wall will increase at a less rate than proportionally with the square of the total height because of the very appreciable thickness of the wall at the top. The width of the base will increase practically proportionally with the height. If therefore W is the weight of the wall to ground level, b its breadth, and H its height there; and W_1 is the weight of wall down to the foundation, b_1 its breadth there, and H_1 the total height—

$$W_1 < W\left(\frac{H_1}{H}\right)^2$$

and

$$b_1 = b\frac{H_1}{H},$$

$$\therefore \quad \frac{W_1}{b_1} < \frac{W}{b}\frac{H_1}{H}.$$

If these be assumed to be equal we shall be taking the maximum compression on the foundation—viz. $\frac{2W_1}{b_1} = \frac{2W}{b}\frac{H_1}{H}$, and the result will be on the safe side. Now $\frac{2W}{b} = 4710$,

$\therefore \frac{2W_1}{b_1} = 4710 \times \frac{20+d}{20}$, where d is the depth from the ground line to foundation level. Then from equation (1)

$$d \times 4{\cdot}7^2 = \frac{p}{w} = \frac{4710}{110} \times \frac{20+d}{20}$$

$$\therefore \quad d \times 22{\cdot}1 = 42{\cdot}8\left(1 + \frac{d}{20}\right) = 42{\cdot}8 + 2{\cdot}1\,d.$$

$$\therefore \quad 20\,d = 42{\cdot}8, \text{ or } d = 2{\cdot}1 \text{ feet.}$$

FOOTINGS

When a block of concrete is placed at the base of a structure in order to spread the weight of the wall or pier, etc., over the

foundation, in order to reduce the intensity of pressure upon it (Fig. 136) the concrete where it projects beyond the wall acts as a cantilever, the upward reactions of the foundation causing tension along the surface D B and inducing a shearing stress in vertical planes between B and D. If the concrete is stepped the steps should fall entirely outside the line B C, or else sections weaker to resist shear than D C would be introduced. The thickness of the concrete block has to be proportioned so as to reduce the maximum tensile stress along D B to a safe limit, unless steel rods are embedded in the concrete near the lower surface to take up the tension; also the maximum intensity of shearing stress must be limited to the working intensity fixed upon, having regard to the fact that the maximum intensity of shearing stress at the axis of the section is one and a half times the average over the section. As an example, take the downward pressure on, and therefore the upward reaction of, the foundation as $2\frac{1}{2}$ tons per square foot, and the projection D B as 2 feet; find what thickness the footing must be (1) to limit the maximum intensity of shearing stress to 3 tons per square foot, and (2) to limit the maximum intensity of tensile stress to 3 tons per square foot.

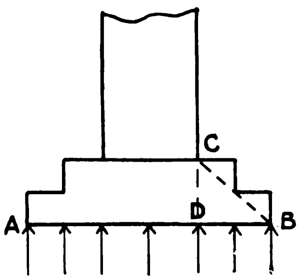

FIG. 136.

Call the thickness of the concrete footing x.

(1) The maximum intensity of shearing stress $= \frac{3}{2} \times \frac{2 \times 2\frac{1}{2}}{x}$

$= 3$ tons per square foot. $\therefore\ x = 2\frac{1}{2}$ feet.

(2) The maximum intensity of tensile stress at D

$$= f = \frac{M}{I} y, \text{ where } M = 2 \times 2\tfrac{1}{2} \times 1,\ I = \frac{x^3}{12}, \text{ and } y = \frac{x}{2}.$$

$$\therefore \quad f = \frac{2 \times 2\frac{1}{2} \times 12}{x^3} \times \frac{x}{2} = 3 \text{ tons per square foot.}$$

$$\therefore \quad x^2 = 10 \text{ or } x = 3{\cdot}2 \text{ feet.}$$

It is thus seen that with a projection of 2 feet and a pressure of $2\frac{1}{2}$ tons per square foot on the foundations a thickness of 3·2 feet at the edge of the wall is required to limit the tensile stress to 3 tons per square foot; the maximum intensity of shearing stress would then be 2·4 tons per square foot, but

the thickness might be limited to the $2\frac{1}{2}$ feet, giving the safe working intensity of shear of 3 tons per square foot fixed upon, or even less, if steel rods are introduced near the lower surface to take up the tension.

Unless the number of piers for a viaduct are limited by the impossibility of finding more than a limited number of positions where it is possible to obtain a foundation, the question arises as to what would be the most economical number of piers, and therefore of spans, to adopt. The following consideration shows that it is not advisable to introduce an additional pier unless its cost is less than that of the main girders and lateral stiffening in a span. The cost of a bridge is generally nearly proportional to its weight. Call the total length of the bridge L, and n the number of spans supposed to be of equal length, then $\frac{L}{n}$ is the length of a single span of the bridge, and the total weight W of one span may be expressed as follows :—$W = \frac{aL}{n} + b\left(\frac{L}{n}\right)^2$, where a and b are constants. $\frac{aL}{n}$ represents the weight of the track and floor system, and $\frac{bL^2}{n^2}$ that of the main girders and lateral bracing. Let A be the cost of the abutments, B the cost of each intermediate pier (supposing the costs to be the same), c the cost per ton of the bridge superstructure, and C the total cost of construction. Then

$$C = A + B(n-1) + cn\left(a\frac{L}{n} + \frac{bL^2}{n^2}\right)$$
$$= A + B(n-1) + c\left(aL + \frac{bL^2}{n}\right).$$

To find what value of n makes C a minimum, differentiate C with respect to n and equate to zero—*i.e.*

$$\frac{dC}{dn} = 0, \text{ or } B - \frac{cbL^2}{n^2} = 0, \text{ or } B = cb\frac{L^2}{n^2}$$

—*i.e.* when the cost of a pier equals the cost of the main girders and lateral stiffening in a span. The economic number of spans therefore is $n = L\sqrt{\frac{cb}{B}}$.

CHAPTER XIV

MASONRY DAMS—STRAIGHT AND CURVED

SEEING that retaining walls are generally of comparatively small dimensions, and that the compressive stresses involved in them are not of great intensity, the primary consideration in their design, as has been pointed out in Chapter XII., is to so arrange that there shall be no tensile stress present in them. The intensity of shearing stress is small, and therefore it is not generally necessary to modify the design to accommodate it.

In the case of low masonry dams for holding up water, the same considerations apply, but in this case, since the horizontal pressure exerted by the water is greater than that exerted by earth, a wider base is necessary to avoid tensile stresses on horizontal planes, and it is very desirable, particularly in higher structures of this nature, to investigate the kind and distribution of the stresses on different planes.

Since a masonry dam is fixed to its base, that fact alone introduces a discontinuity on that plane; moreover, the pressure of the water, generally speaking, suddenly ceases to act at this level, which circumstance still further emphasises the discontinuity. It is reasonable, therefore, to expect that the arrangement of the stresses on and in the neighbourhood of the base will be different from the arrangement of the stresses at higher levels. Experiments carried out on plastic models prove this to be the case.*

If the base is designed of such a width that the centre of pressure on it, when the dam is subjected to water pressure, is at one-third the width of the base from the outer toe, and if the stresses be assumed to be uniformly varying, they would be zero at the inner toe and twice the average value at the outer toe. The resultant force acting on the base is of course inclined, and, as will be proved shortly, would be in such a case parallel to the outer profile of the dam, and if the reacting stresses on the base are all parallel to this resultant force, both the intensity

* "Stresses in Masonry Dams," Sir J. W. Ottley, K.C.I.E., and A. W. Brightmore, D.Sc. *Minutes of Proceedings, Inst. C.E.*, Session 1907-8.

of normal and shearing stress on the base would vary from zero at the inner toe to a maximum equal to twice their respective mean values at the outer toe. On horizontal sections above the level of the base experiment indicates that this is actually the distribution of the stress, but on the base itself, owing to the "fixing" to the foundations there, a different arrangement of stress is introduced. It will be more convenient to consider (1) the distribution of normal stress on the base, and (2) the distribution of shearing stress upon it.

(1) *The distribution of normal stress on the base of the dam.* The uniformly varying distribution of the normal stress is interfered with by the fact that, owing to the triangular shape of the dam, the portions of it near the outer toe are least rigid, so that the most flexible portion of the dam is subjected to the most intense normal stresses. When the height of the structure is great, this leads to a slight upward deflection as the outer toe is approached, which causes the intensity of pressure there to be less than if the dam were perfectly rigid. Thus, instead of the intensity of normal stress in such a case being represented by a straight line A F (Fig. 137), starting at the inner toe and sloping so as to attain a maximum value at the outer toe, the actual line representing this intensity of stress is more horizontal near the outer toe, and shown by the curved line A G. Now if a unit length of the dam at right angles to the paper be considered, and W represents its weight, B F $= \frac{2\,W}{A\,B}$, because the area of the triangle A B F is equal to the weight of the dam—*i.e.* to the total normal pressure on the base; and it follows that if, owing to the flexibility of the dam near the outer toe, the intensity of stress near it is reduced that (*a*) the area enclosed by the vertical line B F and the curve A G, representing the actual intensities of normal stress, must still be equal to W.

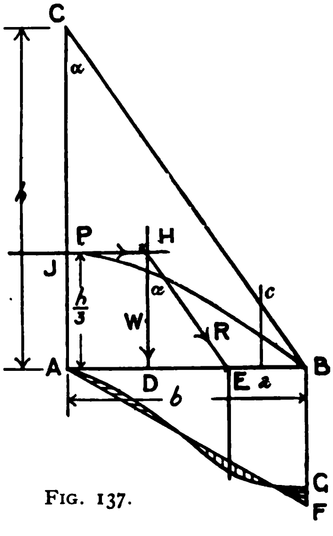

FIG. 137.

Again, since E (Fig. 137) is the centre of pressure or the centre

of resistance, according as we consider the stresses acting on the base or the equal and opposite reactions at the base, the moments of the area of the figure A G B about a vertical line through E must equal zero. Also, since E is at a distance from B equal to one-third A B, the vertical line through E passes through the centre of gravity of the triangle A B F, therefore the moment of the area A B F about the vertical line through E also equals zero. Thus we may express this condition by saying (*b*) that the moment of the areas between the straight line A F and the curve A G about the vertical line through E must vanish; these areas are shown shaded in the figure. There is a third condition which fixes the position of the point G on the curve A G. This is seen if a small triangular wedge of unit length perpendicular to the paper be cut off near B by a vertical line *a c* (Fig. 137), and the forces which keep it in equilibrium vertically be considered. If we call the intensity of shearing stress on *a c*, p_t and the intensity of stress on *a* B, represented by B G in Fig. 137, p_n, the vertical forces acting on this wedge are $p_t \times \overline{ac}$ acting downwards, $p_n \times \overline{a\mathrm{B}}$ acting upwards, and the weight of the triangular wedge—viz. $\frac{\overline{ac} \times \overline{a\mathrm{B}}}{2} \times \rho\, \mathrm{G}$, where ρ is the specific gravity of the masonry of the dam and G is the density of water.

Now p_t and p_n are both finite quantities, therefore the first two forces contain only one factor $\overline{ac}$ and $\overline{a\mathrm{B}}$ respectively, which becomes infinitely small as $\overline{ac}$ approaches B. But the weight contains two factors which become infinitely small in such a case, therefore the weight of the wedge becomes negligible in the limit, and since the triangular wedge is in equilibrium, $p_t \times \overline{ac} = p_n \times \overline{a\mathrm{B}}$; in other words (*c*) B G equals the intensity of shearing stress in a vertical plane near the outer toe multiplied by the ratio of height to base of the dam. It will be noticed that the curve A G, whose ordinates represent the true normal stress on the base, falls inside the straight line A F near the extremities and outside of it in the neighbourhood of the vertical through E, and has therefore a point of inflexion between A and that vertical.

(2) *Distribution of shearing stress on the base of the dam.* Experiments on plastic models indicate that the intensity of shearing stress on the base is more or less constant if the material of which the model is made is homogeneous. Apart from the "fixing" of the dam at the base, the shearing stress would tend

to be distributed more or less in the same manner as the normal intensity of stress—*i.e.* varying from zero at the inner toe to a maximum towards the outer toe, as is the case at levels above the base. But the effect of fixing the base has an opposite result, for an element of width of the dam at A B is compressed by the pressure of the water acting on it at A in the direction A B; the point A therefore tends to move towards B a distance equal to the total amount that the length A B would compress under the action of the water pressure, and points nearer to B would tend to move a less amount in proportion to their reduced distance from B. Since these tendencies to move are resisted by the fixing, shearing stresses are thereby produced, which are a maximum at the inner toe, and gradually reduce to zero at the outer toe. It is seen, therefore, that whilst the vertical forces acting tend to cause the shearing stress to vary from zero at A to a maximum towards B, the horizontal forces acting tend to cause it to be a maximum at A, gradually reducing to zero at B, and the combined result of these two effects is to make the intensity of shearing stress on the base practically constant. But at higher levels, as the effect of fixing becomes less and less felt, the tendency to vary from zero at the face to a maximum at the outer profile predominates.

Intensity of stress normal to vertical planes at points along the base. The variation of the intensity of normal stress on the horizontal plane A B has been considered, also the distribution of shearing stress along A B, and it is known that the intensity of shearing stress at any point on the vertical plane through it will be equal to its intensity on the horizontal plane there; but in order to obtain a complete idea of the arrangement and kind of stress on the different planes at any point along A B, it still remains to obtain an idea of the variation of the normal intensity of stress on vertical planes at any point along the base. At the face A C of the dam the horizontal pressure of the water acts at J, one-third the height of the water above A, supposing the water to reach the top of the dam. Again, if we consider the small triangle a B c near the outer toe (Fig. 137), we see that for its horizontal equilibrium the intensity of normal stress on $a\,c$ ($p_n{}^1$) multiplied by $\overline{a\,c}$ equals the intensity of shearing stress on a B multiplied by $\overline{a\,\text{B}}$ —that is

$$p_n{}^1 \times \overline{a\,c} = p_t \times \overline{a\,\text{B}}.$$

Also we have seen that $p_n \times \overline{a\,\text{B}} = p_t \times \overline{a\,c}$,

$$\therefore \quad \frac{p_n{}^1 \times \overline{a\,c}}{p_t \times \overline{a\,c}} = \frac{p_t \times \overline{a\,\text{B}}}{p_n \times \overline{a\,\text{B}}}.$$

The term on the left side of the equation is the cotangent of the angle of inclination to the horizontal of the resultant force on $a\,c$, and the term on the right side is similarly the cotangent of the angle of inclination to the horizontal of the resultant force on $\overline{a\,\text{B}}$, therefore these two forces have the same inclination, and must act in the same straight line and be parallel to B C, since there can be no resultant stress on B C. Now the resultant force on $\overline{a\,\text{B}}$ acts through its centre, therefore the resultant force on $a\,c$ also acts through its middle point. We now know three points on the curve which is the locus of the point of application of the resultant horizontal force across vertical planes—viz. J where the curve will be tangent to a horizontal line, the middle point of $a\,c$ and B. The curve J B is drawn in Fig. 137; at J the point of application is at one-third the height and at a it is at half the height of the vertical. Now the resultant horizontal force on any vertical section is proportional to the distance of that section from B, because it equals the total horizontal shear between that section and B, and it has been shown that the intensity of that shear is practically constant. Also the area of the vertical section (of unit length) varies as the distance of the vertical section from B; consequently the *average* intensity of horizontal stress on all vertical sections is constant. But, as we have just seen, the point of application of its resultant at the inner face is at one-third the height from the base, whilst at the outer toe it is at one-half the height from the base, therefore the actual intensity of normal stress $p_n{}^1$ on vertical planes at points along B A increases from B to A.

It was shown in Chapter VI., page 116, that if at any point $p_n\,p_n{}^1 > p_t{}^2$, the principal stresses at that point are both of the same sign; and the above equation shows that at B, $p_n\,p_n{}^1 = p_t{}^2$, and Fig. 137 shows that p_n tends to slightly increase for some distance from B, and it has just been proved that $p_n{}^1$ increases from B to A, therefore for some distance from B towards A, $p_n\,p_n{}^1 > p_t{}^2$; consequently, both principal stresses are in this case compressions for that distance, and there will be no tension on any plane through such points. At some point between A and B, $p_n \times p_n{}^1$ will again equal $p_t{}^2$, and from that point onwards towards A there will be a tensile stress on certain

planes. The inclination of the plane of maximum tension at this point, where the tension begins to manifest itself, is found from the formula on page 116; viz. $\tan \theta = \frac{p_n^1 - p}{p_t}$. At the point in question the tension equals zero, so that the inclination of the normal to the plane of zero stress is $\tan^{-1} \frac{p_n^1}{p_t}$. Now at B also, the principal stress normal to the maximum stress equals zero and $\frac{p_n^1}{p_t} = \frac{AB}{AC}$. p_n^1 is less than p_t for some distance from B, but by the time the point in question is reached p_n^1 will be more nearly equal p_t, consequently the normal to the plane there is inclined to the horizontal at an angle approaching 45 deg., and the plane itself is inclined at an angle of about 45 deg., measured from A B downwards. As the tension increases $\tan \theta$ increases, and the plane of maximum tension becomes gradually less inclined to A B. At A, *considering a point in the foundation*, both p_n and p_n^1 are zero, so that the maximum tension and compression will be equal to p_t, and the plane of maximum tension will be inclined at an angle of 45 deg. below A B. But at a point near A, in the dam *above the base*, p_n^1 has its maximum value, so that the plane of maximum tension in the dam itself is inclined at a smaller angle to the horizontal. The intensity of shearing stress in vertical planes near the outer toe will be greater at some distance above the base than at the base, because, as has been pointed out, the intensity of shearing stress on the horizontal plane is practically constant on the base, but at higher levels it varies from zero at the inner face to twice its average value on that plane at the outer profile; but it will not exceed the value that would be obtained for it on the assumption that the shearing stress on the base is proportional to the normal intensity of stress at each point—*i.e.* if the same assumption is made at the base as at higher levels.

The effect of the foregoing considerations is to show that if the pressures and reactions on the base are assumed to be uniformly varying, and to act parallel to the resultant force on the base, that the intensities of stress allowed for in the calculations on this assumption would be greater than the actu intensity of stress developed in the dam, both as regards co pressive and shearing stresses.

This being the case, a dam may be safely designed o

assumption. The conditions to be satisfied by the design are therefore :—

(1) There shall be no tension on horizontal planes in the dam—*i.e.* the resultant pressure reservoir, full or empty, must fall within the middle third of horizontal sections.

(2) The maximum intensity of compressive stress on any plane must not exceed the permissible working intensity, generally 8 to 12 tons per square foot.

(3) The maximum intensity of shearing stress on either the horizontal or vertical plane must not exceed a safe limit; the maximum intensity of shearing stress is really fixed when the working intensity of compressive stress is, because, as will be shown, the maximum intensity of shearing stress equals nearly one-half the maximum intensity of compressive stress.

(4) The structure must be impervious to water.

Design of "Low" Dam

(1) It is readily seen that a triangular section with a vertical face next to the water and having a certain proportion of width of base to height, satisfies the first of these conditions. When there is no water against the dam the only force acting on the base is the weight, and as this acts through the centre of gravity, cutting the base at one-third its width from the face, this condition is satisfied. Moreover, when the reservoir is full of water, if the width of the base is equal to the height divided by the square root of the specific gravity of the masonry, it will be seen that the condition is also satisfied, for in Fig. 137 call h the height and b the breadth of base, and let ρ be the specific gravity of the masonry, and let $b = \dfrac{h}{\sqrt{\rho}}$.

Let the resultant of the pressure of the water and the weight of the dam cut the base A B in E, and the line through the centre of gravity cut it at D. The resultant pressure of the water $= \mathrm{P} = \dfrac{\mathrm{G}\,h^2}{2}$, where G is the density of water, and this force is applied at a height $\dfrac{h}{3}$ above the base.

The weight of the dam of unit length

$$= \frac{h}{2} \times b \times \rho\,\mathrm{G} = \frac{h^2}{2}\sqrt{\rho}\,\mathrm{G}.$$

Now $\frac{P}{W} = \frac{DE}{HD}$, substitute for P and W, and write $HD = \frac{h}{3}$.

$$DE = \frac{h}{3} \times \frac{G h^2}{h^2 \sqrt{\rho}\, G} = \frac{h}{3\sqrt{\rho}} = \frac{b}{3}.$$

Thus $DE = DA = \frac{b}{3}$, $\therefore$ $BE = \frac{b}{3}$, and the resultant force on the base acts at one-third the width of the base from the outer toe. Thus a triangular section of dam with a vertical face satisfies the first condition laid down.

(2) With regard to the second condition, it is obvious that with the reservoir empty the maximum intensity of compression will occur at the inner toe, and will equal $\frac{2W}{b}$, and with the reservoir full the maximum intensity of compression will occur at the outer toe, and if it is assumed that the reacting stresses are parallel to the resultant force on the base, and are uniformly varying, which assumption, it has been shown, gives results which are rather greater than the actual stresses induced, the maximum intensity of compression on the horizontal plane equals $\frac{2R}{b}$, but its actual maximum intensity would be on a plane at right angles to the resultant, where its value is $\frac{2R}{b\cos\alpha}$. Since $R = \frac{W}{\cos\alpha}$, the maximum intensity of compressive stress on this plane may be written $\frac{2W}{b\cos^2\alpha}$. The actual maximum stress will be rather less than this for the reasons already given. It is clear that the maximum height to which this triangular section can be used will be attained when the maximum compressive stress in it equals the permissible working intensity of compression for the material employed. If this working intensity be denoted by s, the maximum height can be deduced by equating it to the above expression, as follows :—

$$\frac{2W}{b\cos^2\alpha} = s.$$

Where $W = \frac{bh}{2}\rho G$, and $\cos\alpha = \frac{h}{\sqrt{b^2 + h^2}}$.

Substituting, $\frac{bh\rho G}{bh^2}(b^2 + h^2) = s.$

Taking $b = \frac{h}{\sqrt{\rho}}$, we get $h = \frac{s}{G(1 + \rho)}$.

$G = \frac{1}{36}$ ton, and if s be taken as 10 tons per square foot and

$\rho = 2\frac{1}{4}$, $h = \frac{360}{3\frac{1}{4}} = 110$ feet.

In dams of greater height, the width below this level has to be increased at a greater rate than in the triangular section, in order to keep the maximum intensity of compression within the desired limit. The question of the proper width for greater heights will be reverted to later.

(3) The fixing of the maximum intensity of compression also determines the maximum intensity of shearing stress in the dam, because it was shown in Chapter VI. that the maximum intensity of shearing stress is half the difference of the principal stresses. Therefore, if one principal stress near the outer toe = 10 tons per square foot when the other principal stress is zero, the maximum intensity of shear in planes inclined to it at 45 deg. would be $\frac{10}{2} = 5$ tons per square foot. It would really be less than this, because it has been shown that the 10 tons per square foot in compression is not actually attained. Over the base, however, the shearing stress, as has been stated, is of more or less constant intensity, having an average value of $\frac{G h^2}{2 b} = \frac{G h \sqrt{\rho}}{2} = \frac{110}{36} \times \frac{3}{4} = 2{\cdot}3$ tons per square foot for the dam of limiting height.

(4) The compliance with this condition is a matter of proper selection of the materials used and of sound methods of construction.

To obtain a closer value for the actual principal stress near the outer toe. The normal intensity of stress on the base near the outer toe for the above dam $= 10 \times \cos^2 \alpha = 10 \times \frac{9}{13}$ $= 6{\cdot}9$ tons per square foot.

The intensity of shear, if constant, has been shown above to be 2·3 tons per square foot. The intensity of pressure normal to a vertical plane $p_n{}^1$, which is then equal to the average value of the normal pressure on that plane, since its resultant acts at one-half the height, $= \frac{G h}{2} = 1{\cdot}5$ tons per square foot.

From Chapter VI., page 116, the maximum principal stress

$$= \frac{p_n + p_n^1 + \sqrt{(p_n - p_n^1)^2 + 4p_t^2}}{2}$$

$$= \frac{8{\cdot}4 + \sqrt{(5{\cdot}4)^2 + 4 \times (2{\cdot}3)^2}}{2} = \frac{8{\cdot}4 + 7{\cdot}1}{2}$$

$= 7{\cdot}8$ tons per square foot.

This is seen to be considerably less than the 10 tons, but as the intensity of shearing force increases immediately above the base it is probable that near the outer toe the intensity of shear, even at the base, is greater than the average. This would cause p_n^1 to also have a greater value.

Next find the maximum intensity of tension in a dam of the above height at the inner toe, (1) *below the level of the base,* (2) *above that level.*

Note.—The horizontal section at the level at which the water pressure ceases to act is called the base.

(1) At the inner toe we have $p_n = 0$, and below the level of the base $p_n^1 = 0$, thus we have on both horizontal and vertical planes there, only a shearing stress of 2·3 tons per square foot; the plane of maximum tension will therefore be inclined down from the horizontal at an angle of 45 deg., and the intensity of tension on it will be 2·3 tons per square foot, because the ellipse of stress becomes a circle.

(2) Above the level of the base near the inner toe $p_n = 0$, as before, but p_n^1 now equals $\text{G}\,h = 3{\cdot}1$ tons per square foot.

From Chapter VI. the maximum tension

$$= \frac{p_n^1 - \sqrt{(p_n^1)^2 + 4p_t^2}}{2} = \frac{3{\cdot}1 - \sqrt{9{\cdot}6 + 21{\cdot}2}}{2}$$

$$= \frac{3{\cdot}1 - 5{\cdot}5}{2} = -1{\cdot}2 \text{ tons per square foot.}$$

And the inclination of the normal to the plane is given by

$$\cot\theta = -\frac{p}{p_t} = \frac{1{\cdot}2}{2{\cdot}3} = \tfrac{1}{2},$$

so that the plane itself is inclined downwards from the horizontal at an angle $\cot^{-1} 2{\cdot}0$.

This shows that the maximum tension in the dam itself, above the base, is considerably less, and the plane on which it acts is less inclined to the horizontal, than below the level where the water pressure ceases to act, and since the shearing stress at the inner face will extend only a short distance above the inner toe, the tension there is not of much consequence, especially

as selected materials are always used in this position for the sake of securing water-tightness. The tension below the base line cannot be eliminated, but its plane of action can be made steeper by anchoring that part of the foundation to the rock in front of it, if present, which would have the effect of causing p_n^1 to be a tension instead of being zero. It is interesting to inquire how much wider the base of the dam would have to be to entirely eliminate tension above the base.

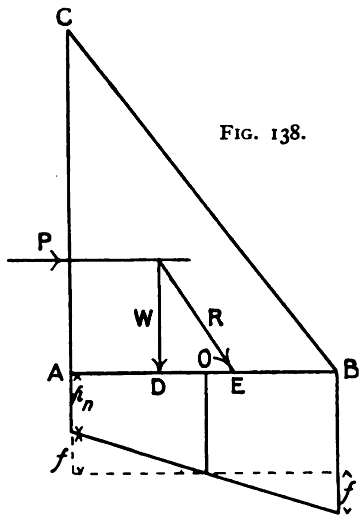

FIG. 138.

To find the width of base necessary to avoid tension on any plane in the dam above the level of the base.

We have $p_t = \frac{P}{b} = \frac{G h^2}{2 b}$.

At A $$p_n^1 = \frac{2 P}{h} = \frac{2 G h^2}{2 h} = G h.$$

To avoid tension at A in the dam, $p_n \times p_n^1$ must be at least equal to p_t^2 there. That is—

$$p_n \times G h = \frac{G^2 h^4}{4 b^2}, \text{ or } p_n = \frac{G h^3}{4 b^2}.$$

The moment M, which increases the normal pressure at B and reduces that at A by an amount f, equals W $\times$ x, where x = O E (Fig. 138), where

$$W = \rho G \times \frac{b \times h}{2}, \text{ and } x = DE + AD - AO,$$

or $$x = \frac{h}{3} \times \frac{P}{W} + \frac{b}{3} - \frac{b}{2} = \frac{h}{3} \times \frac{P}{W} - \frac{b}{6}.$$

$$\therefore \quad M = W \times x = \frac{h}{3} \times P - \frac{b}{6} W = \frac{G h^3}{6} - \rho \frac{G b^2 h}{12}.$$

Now, $M = I \times \frac{2 f}{b}$ (1), where I, the moment of inertia about an axis through the centre of A B perpendicular to the paper,

$$= \frac{b^3}{12} \text{ and } f = \frac{W}{b} - p_n = \rho \frac{G h}{2} - \frac{G h^3}{4 b^2}.$$

Substituting in (1)

$$\frac{G h^3}{6} - \rho \frac{G b^2 h}{12} = \frac{b^2}{12}\left(\rho G h - \frac{G h^3}{2 b^2}\right),$$

or
$$h^2 = b^2 \rho - \frac{h^2}{4}.$$

$$\therefore \qquad b = \sqrt{\frac{5}{4}} \times \frac{h}{\sqrt{\rho}} = 1{\cdot}12 \frac{h}{\sqrt{\rho}}.$$

It is thus seen that to avoid tension altogether in the dam the base would have to be made nearly one-eighth as wide again as its value when calculated to avoid tension on the base A B at A, and there would still be a tension in the foundation below the level of the base.

A simple calculation serves to prove that the shearing stress is not constant on horizontal planes above the base; the total shear on such planes is reduced in proportion to the square of the depth of the planes in question below the crest, but the area of the horizontal section decreases with the depth, therefore the average intensity of shearing stress on such planes varies as the depth below the crest. It is easy to prove that the average intensity of shearing stress in a vertical plane near the outer toe is greater than the constant shearing stress on the base, showing that near the outer profile the shearing stresses increase, instead of decreasing, at levels above the base, showing that the shear is not uniformly distributed on such planes, but is a maximum near the outer profile. Take the case of the dam 110 feet high, and find the total shearing stress on a vertical section $a\ c$, 10 feet from the outer toe (Fig. 137). The height of this section equals $10\ \sqrt{\rho} = 15$ feet.

The total shear on $a\ c$ equals the upward normal reaction on B a minus the weight of the triangular wedge a B c. The normal reaction at B has been shown to be 6·9 tons per square foot, and assuming that it is uniformly varying at c it would be equal to $6{\cdot}9 \times \frac{95}{110} = 6$ tons per square foot.

The total normal reaction on B a therefore equals

$$\frac{6{\cdot}9 + 6{\cdot}0}{2} \times 10 = 64{\cdot}5 \text{ tons}.$$

The weight of the wedge a B c equals

$$\frac{15 \times 10}{2} \times \frac{9}{4} \times \frac{1}{36} = 4{\cdot}7 \text{ tons}.$$

Therefore the total shear on $a\ c = 64{\cdot}5 - 4{\cdot}7 = 59{\cdot}8$ tons.

The average intensity of shearing stress on $a\,c = \frac{59{\cdot}8}{15} = 4$ tons per square foot, which is considerably greater than the average shear on the base—viz. 2·3 tons per square foot, showing that on planes above the base the shearing stress cannot be uniformly distributed, but its intensity must be a maximum near the outer profile.

The crest of a dam in actual practice does not come to a point as shown in Figs. 137 and 138, but has to have a definite thickness to enable it to resist the wave action of the water in the reservoir and shocks from floating bodies, and very often to provide a roadway across the dam. The addition of a triangular mass of masonry A B C (Fig. 139) causes the line through the centre of gravity of the masonry above any horizontal section to fall within the middle third down to a depth shown by the section D E, where $\frac{1}{3}$ D E $= \frac{2}{3}$ A B, $\therefore$ D E $= 2$ A B and A D $= 2$ B C. If we denote A B by a, then A D $= 2\,a\,\sqrt{\rho}$.

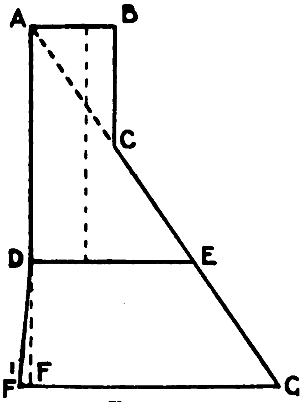

FIG. 139.

At sections below D E the line through the centre of gravity of the triangle A B C will fall in the *inner* third of the section, therefore the line through the centre of gravity of the masonry above such sections falls outside the middle third by a distance whose ratio to the breadth at the section continues to increase as depth increases down to a certain level F G, when this ratio commences to decrease again. If, therefore, the width is increased at the level F G by an amount F F^1 so that the vertical line through the centre of gravity of the masonry above F^1 G passes through the point distant one-third F^1 G from F^1, and the face is battered from D to F^1, the resultant with the reservoir empty will be prevented from falling outside the middle third and below F^1 the face can be made vertical again down to the depth $\frac{s}{G\,(1+\rho)}$.

To find the depth A F (Fig. 139) when the vertical line through

$\therefore$ the centre of gravity of the figure A B C G F is distant from F a length, c, such that the ratio c : F G is a minimum.

Let A F $= h$, then F G $= \dfrac{h}{\sqrt{\rho}}$.

Taking moments about F,

$$\frac{a^2 \times \sqrt{\rho}}{2} \times \tfrac{2}{3}\, a + \frac{h^2}{2\sqrt{\rho}} \times \tfrac{1}{3}\, \frac{h}{\sqrt{\rho}} = c\left(\frac{a^2 \times \sqrt{\rho}}{2} + \frac{h^2}{2\sqrt{\rho}}\right),$$

$$\therefore \qquad c = \frac{2\, a^3\, \rho^{\frac{3}{2}} + h^3}{3\,(a^2\, \rho^{\frac{3}{2}} + h^2\, \rho^{\frac{1}{2}})},$$

and $\dfrac{c}{h}$ is a minimum when $\dfrac{d}{d\,h}\, \dfrac{2\, a^3\, \rho^{\frac{3}{2}} + h^3}{3\,(a^2\, \rho^{\frac{3}{2}}\, h + h^3\, \rho^{\frac{1}{2}})} = 0$

—*i.e.* when $h = 3{\cdot}1\, a\, \sqrt{\rho}$.

The length of the offset F F¹ can be calculated by taking moments about F as before, but taking the triangle F D F¹ into account, and remembering that the vertical line through the centre of gravity of the figure F¹ D A B C G passes through the point distant $\dfrac{\text{F}^1\,\text{G}}{3}$ from F¹.

Design of "High" Dam

When the dam is higher than 110 feet with the constants taken, the breadth of the base has to be made wider than given by the triangular section, and an expression for the breadth at any depth below the crest is given below, which involves, however, the weight of the dam down to that section, but in a term which is only about one-seventh the value of the expression for the square of the breadth. This enables the thickness of the dam at sections below 110 feet to be calculated at depths increasing by say 20 feet at a time. A value for the weight down to the new depth can be found as a first approximation by continuing the lines of the profile in the last 20 feet thickness, and the breadth calculated on the basis of this weight. A more exact value for the weight can then be obtained by substituting this breadth in its calculation, from which a corrected value of the breadth can be obtained by applying the formula as before.

In obtaining this formula it has to be borne in mind that the resultant force on these horizontal sections with the reservoir full will no longer act at one-third the width from the outer profile, but its point of action, the centre of pressure, will fall at a point inside the middle third. It is necessary, therefore,

first of all to find an expression for the distance of the centre of pressure from the outer profile, in terms of the breadth of the dam at the section, the maximum intensity of stress and the average intensity of stress on the section. Call b the breadth

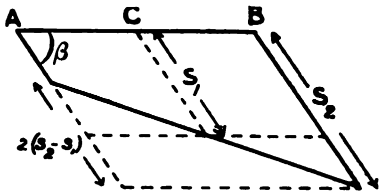

FIG 140.

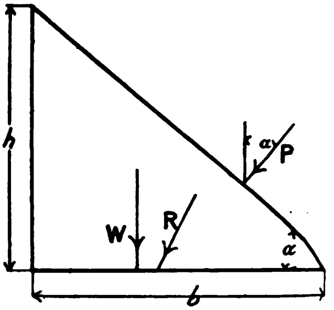

FIG. 141.

at the section, s_1 the average intensity of stress on the section, s_2 the maximum intensity of stress on the section, and c the distance of the centre of pressure from the outer profile. Let y denote the stress at any distance x from B, then from Fig. 140 it is clear that $y = s_2 - x\,\dfrac{2\,(s_2 - s_1)}{b}$.

Now the moment of the total stress on A B about B $= c\,b\,s_1 \sin^2\beta$,

$$\therefore\ c\,b\,s_1 = \int_0^b x\,y\,dx = \int_0^b s_2\,x\,dx - \int_0^b \frac{2\,(s_2 - s_1)}{b}\,x^2\,dx$$

$$= \tfrac{1}{2}\,s_2\,b^2 - \frac{2\,(s_2 - s_1)}{b} \times \frac{b^3}{3}$$

$$= \tfrac{2}{3}\,s_1\,b^2 - \tfrac{1}{6}\,s_2\,b^2$$

$$\therefore\quad c = \frac{b}{3}\left(2 - \frac{s_2}{2\,s_1}\right).$$

In order to obtain a workable expression for b, it is assumed that the vertical line through the centre of gravity of the masonry above the section and *of the water over the inner face* cuts b at a distance equal to $\dfrac{b}{3}$ from the inner face. The effect of this stipulation is to cause the centre of pressure when the reservoir is empty to be slightly inside the middle third, but since the necessary batter of the inner face is very slight, the weight of

the water over it is small, so that the centre of gravity of the masonry practically falls at one-third the distance from the inner profile. Let W be the weight of the masonry above the section considered and of the water over the inner face; the resultant horizontal pressure of the water, $P = \frac{G h^2}{2}$; and, the resultant force on the horizontal section, $R = \sqrt{W^2 + P^2}$. If s is the maximum intensity of stress in the dam,

$$s_2 = s \cos \alpha = s \frac{W}{\sqrt{W^2 + P^2}}.$$

The average intensity of stress on b is s_1, and

$$s_1 = \frac{\sqrt{W^2 + P^2}}{b},$$

$$\therefore \quad c = \frac{b}{3}\left(2 - \frac{s_2}{2 s_1}\right) = \frac{b}{3}\left(2 - \frac{s\, b\, W}{2\,(W^2 + P^2)}\right).$$

Now the breadth b

$$= \frac{b}{3} + \frac{P}{W} \times \frac{h}{3} + c$$

$$= \frac{b}{3} + \frac{P}{W} \times \frac{h}{3} + \frac{b}{3}\left(2 - \frac{s\, b\, W}{2\,(W^2 + P^2)}\right),$$

$$\text{hence } \frac{P h}{W b} = \frac{s\, b\, W}{2\,(W^2 + P^2)},$$

$$\therefore \quad b^2 = \frac{2\, P\, h\,(W^2 + P^2)}{s\, W^2}.$$

Substituting for P, $b = \sqrt{\frac{w\, h^3}{s}\left(1 + \frac{w^2 h^4}{4 W^2}\right)}$,

where the term $\frac{w^2 h^4}{4 W^2}$ equals about $\frac{1}{7}$.

This formula enables the necessary breadth of the dam at depths greater than 110 feet (with the data taken) to be determined as explained above. The method of applying the formula is explained in detail with a numerical example in "Waterworks Engineering," page 221.*

In the case of a temporary dam in which timber enters largely into its composition, the slope of the dam may with advantage be turned towards the water (Fig. 141) instead of away from it, as in a masonry dam. The water pressure P will, of course, act normally to the sloping face and equals $\sqrt{h^2 + b^2} \times \frac{G h}{2}$, and the weight W equals $\frac{h b}{2} \rho$ G, where ρ is

* "Waterworks Engineering." Third Edition. Tudsbery & Brightmore.

the specific gravity of the material of the dam. Let x be the distance of the point of intersection of the resultant R with the base from that of W with it. Taking moments about the former point,

$$\frac{h}{3} \mathrm{P} \sin \alpha + \mathrm{W} x = \mathrm{P} \cos \alpha \left(\frac{b}{3} - x\right).$$

Substituting the above values for P and W, and substituting for sin α and cos α

$$h\, b\, x\, \mathrm{G}\, (1 + \rho) = \frac{\mathrm{G}\, h}{3} (b^2 - h^2)$$

or

$$x = \frac{1}{1 + \rho} \times \frac{b^2 - h^2}{3\, b}.$$

To avoid tension, x must be positive because W acts at the distance $\frac{b}{3}$ from A—*i.e.* b = or > h.

It will be noticed that if b > h, x will be greater—*i.e.* the resultant on the base will be nearer the centre—the smaller the density of the material of which the dam is built.

In order to be in a position to ascertain whether—and if so, when—it is advantageous to build a dam curved in plan, the relation between radial and circumferential stresses in thick cylinders will next be considered.

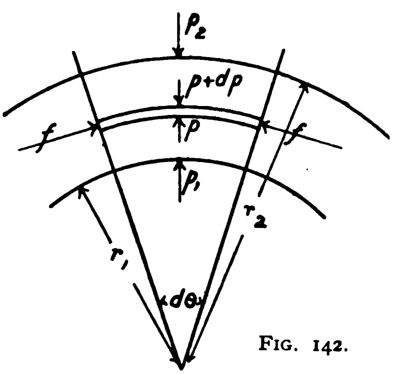

FIG. 142.

Assume that, as there is no stress in the longitudinal direction, all particles in a cross-sectional plane remain in a plane.

Let p be the intensity of radial stress at any point of the cross-section (Fig. 142) and f the intensity of circumferential stress, both f and p act in the plane of the cross-section.

A strain will be produced in the longitudinal direction, which is the lateral strain due to both f and p, if E is Young's modulus of elasticity a stress divided by E is the strain in the line of the stress, and this strain multiplied by $\frac{1}{\sigma}$, where $\frac{1}{\sigma}$ is Poisson's ratio, is the strain in directions at right angles to the stress. There-

fore the longitudinal strain equals $\frac{p+f}{\sigma E}$; as we assume that all particles in a cross-section remain in a plane, this strain is constant for all points in the cross-section, and since σ and E are constant, we have $p + f = 2c$ (1) where c is a constant and the 2 is introduced to simplify the resulting equation. Take a sector of the cylinder (Fig. 142) between two radii subtending an angle $d\theta$ at the centre, and consider the equilibrium of a circular strip radius r and thickness dr in the radial direction. The radial pressure acting on one side of this strip $= p\,r\,d\theta$, and on the other side it equals $p\,r\,d\theta + d\,.\,p\,r\,d\theta$; the difference of the two is $d\,.\,p\,r\,d\theta$. This is balanced by the resolved part in the radial direction of the stress on the two ends of the strip $= 2f\,dr\frac{d\theta}{2} = f\,dr\,d\theta$,

$$\therefore \quad d\,.\,p\,r\,d\theta = f\,dr\,d\theta.$$

Since $d\theta$ is constant, $d\,.\,p\,r = f\,dr$.

Taking the differential of the term on the left side—

$$p\,dr + r\,dp = f\,dr$$

from (1)

$$f = 2c - p,$$

$$\therefore \quad p\,dr + r\,dp = (2c - p)\,dr,$$

or

$$r\,dp = 2(c - p)\,dr;$$

$$\therefore \quad \frac{dp}{p - c} + \frac{2\,dr}{r} = 0.$$

Integrating $\log(p - c) + \log r^2 = \log c^1$.

Taking the antilogarithms $(p - c) \times r^2 = c^1$,

or

$$p = c + \frac{c^1}{r^2} \quad . \quad . \quad . \quad . \quad (2)$$

and

$$f = 2c - p = c - \frac{c^1}{r^2} \quad . \quad . \quad . \quad (3)$$

Case I

If the pressure is applied on the convex side, when $r = r_1$, $p = 0$, and $f = f_1$, and when $r = r_2$, $p = p_2$, and $f = f_2$, by substituting these values in equations (2) and (3)

$$c + \frac{c^1}{r_1^2} = 0, \text{ and } f_1 = c - \frac{c^1}{r_1^2}$$

$$p_2 = c + \frac{c^1}{r_2^2}, \text{ and } f_2 = c - \frac{c^1}{r_2^2}.$$

From which

$$f_1 = -c^1 \times \frac{2}{r_1^2}, \text{ and } p_2 = c^1\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right).$$

$$\therefore \quad f_1 = \frac{2 p_2 r_2^2}{r_2^2 - r_1^2}, \text{ and } f_2 = -c^1\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right),$$

$$\therefore \quad f_2 = p_2 \frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}.$$

Since $p + f$ is constant and $p_1 = 0$, $p_2 + f_2 = f_1$, the expression for f_2 could be obtained from that for f_1 by subtracting p_2 from the value of the latter.

It will be noticed that f_1, the circumferential stress at the inside, is greater than that at the outside.

Case II

It may be mentioned in passing that if the pressure is applied inside, as in the case of a hydraulic pipe or a cannon, the values of f_1 and f_2 are obtained by simply interchanging the suffixes 1 and 2 in the former expressions—*i.e.*

$$f_2 = -\frac{2 p_1 r_1^2}{r_2^2 - r_1^2}, \text{ and } f_1 = -p_1 \times \frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}.$$

It will be noticed that the stresses are now both tensions, and that f_1 has still the larger value, but it is not so great as when the pressure is applied at the outside. To illustrate the application of the latter formulas, let it be desired to find the thickness of a cast-iron pipe 6 inches diameter, if the internal pressure is $\frac{1}{2}$ ton per square inch and the maximum permissible intensity of working stress is 1 ton per square inch.

The maximum tension $= f_1 = -1$ and $p_1 = \frac{1}{2}$, substituting these values in the above equations $1 = \frac{1}{2}\,\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}$—*i.e.*

$$2 r_2^2 - 2 r_1^2 = r_1^2 + r_2^2, \; r_2^2 = 3 r_1^2 = 27.$$

$$\therefore \quad r_2 = 5{\cdot}2.$$

Therefore the necessary thickness $= r_2 - r_1 = 5{\cdot}2 - 3 = 2{\cdot}2$ inches.

Curved or Arched Dams. Let the arch be the segment of a circle in plan. If it be assumed that the thrust of the water normal to the inner—convex—surface of an arched dam is transmitted to the abutments in every horizontal lamina of the structure, according to the principle of action of an arch, the compressive stresses due to the water pressure can be determined from the formula for the thick cylinder.

Consider a horizontal lamina of unit depth at a distance h below the crest (Fig. 143), and call its breadth b, and let the radius of the inside profile there be r_2 and that of the outside

profile r_1. The intensity of radial pressure of the water on the lamina equals G h, and let f be the maximum intensity of circumferential stress which has been already shown to be at the concave or downstream face. From the formula in Case I. above

$$f = \frac{2\,\text{G}\,h\,r_2^2}{r_2^2 - r_1^2}, \text{ or } r_2^2 - r_1^2 = \frac{2\,\text{G}\,h\,r_2^2}{f},$$

$$\therefore \qquad r_1 = r_2\sqrt{1 - \frac{2\,\text{G}\,h}{f}},$$

and

$$b = r_2 - r_1 = r_2\left(1 - \sqrt{1 - \frac{2\,\text{G}\,h}{f}}\right).$$

If the inner face is vertical r_2 will be constant, and this expression for b gives the thickness necessary at any depth h below the crest.

If r_2 be taken as 250 feet, f equal to 10 tons per square foot, and h equal 100 feet, it will be found that b works out to 83 feet. Now for a gravity dam of triangular section, it only requires to be 67 feet; it will be seen therefore that to permit of reducing the section due to using the arched form of dam the radius must be very sharp, unless the working intensity of stress is considerably increased. As an example of the use of the above formula, find the maximum intensity of stress in the well-known Bear Valley dam, which has a radius at the water face of 335 feet, at a depth of 48 feet below the crest where the breadth is 8·5 feet.

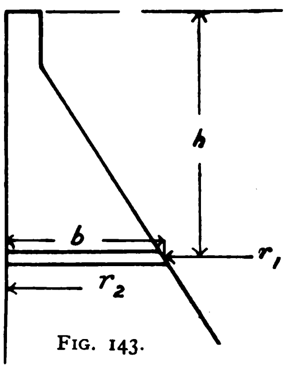

FIG. 143.

The expression deduced for b may be written

$$\left(1 - \frac{b}{r_2}\right)^2 = 1 - \frac{2\,\text{G}\,h}{f}, \text{ or } 2\frac{b}{r_2} - \frac{b^2}{r_2^2} = \frac{2\,\text{G}\,h}{f},$$

from which

$$f = \frac{2\,\text{G}\,h}{\frac{b}{r_2}\left(2 - \frac{b}{r_2}\right)},$$

substituting

$$f = \frac{2 \times \frac{1}{36} \times 48}{\frac{8\cdot5}{335}\left(2 - \frac{8\cdot5}{335}\right)} = 53 \text{ tons per square foot.}$$

It will thus be seen that this dam is subjected to a working intensity of pressure much higher than that usually considered advisable.

CHAPTER XV

CONCRETE STRUCTURES REINFORCED WITH STEEL RODS WHERE IN TENSION

THE practice of using concrete reinforced with steel bars embedded in its mass in order to take up tension in positions where concrete, masonry, or brickwork would be of themselves unsuitable on account of the presence of such tension, has considerably extended of late years. Applications of this form of construction have already been alluded to in the case of concrete footings and certain types of retaining walls. Another very important application for building is for beams, floor slabs, and columns. The steel rods are so placed in the concrete that the former take the tensile stress, whilst the latter takes the compression—*i.e.* the steel must be placed in the positions where the tensile stresses exist.

In the case of a beam of rectangular section supported at the two ends and uniformly loaded, the intensity of tensile stress is a maximum at the under surface at the centre, the plane on which it acts there being vertical; and it gradually diminishes to zero at this surface as the ends are approached. At the neutral plane the tension on vertical planes is everywhere zero, but the intensity of shearing stress is in each section greatest at the neutral plane, being a maximum at the ends and diminishing to zero at the centre. In Chapter VI., page 120, it was shown that if the only stresses acting on two planes at right angles are shearing stresses, there is a tensile stress of equal intensity on a plane making an angle of 45 deg. with these planes; thus at the neutral plane there is a maximum tension of equal intensity to the shearing stress on planes sloping downwards from the neutral plane at an angle of 45 deg. Thus the maximum tension occurs at the central section at the lower surface and acts on a vertical plane, and at the extremities it occurs at the neutral plane on a plane sloping downwards from it at an angle of 45 deg. It is therefore necessary, if the tensile stress is to be taken up by steel rods in addition to embedding rods near to the lower surface, as the distance from the centre of

the beam increases to introduce reinforcement in the direction of the tensile stress due to the shear. If straight rods of uniform cross-section are used for the reinforcement, some of them should be curved upwards as they approach the ends of the beam, and since the stress in them would have a finite value at the ends, it is necessary that they should be properly secured, for instance, by passing them through a flat plate washer and screwing a nut on the threaded ends of the rods, or by splitting open the ends. Instead of curving up the rods towards their ends, stirrups, connected to the rods and inclined upwards beyond the neutral plane, may be employed. It will be noticed that the introduction of the steel rods causes the shearing stress to be constant in horizontal and vertical planes between the rods and the neutral plane; consequently the maximum tensile stresses will be constant at points in vertical planes between the rods and the neutral plane.

The modulus of elasticity for concrete is a varying quantity according to its composition and its manipulation before it is in its final position; it also decreases as the stress upon it increases, but its value is considerably less than that for steel, being about one-fifteenth of its value for this material. One of the most useful properties of concrete, in this connection, is that the concrete in setting contracts, and therefore obtains a good grip on the steel rods if they are round or square in section. This effect, moreover, puts the rods into initial compression to some extent at right angles to their length, so that the rods do not immediately commence to contract in cross-section when the tensile stress comes upon them. It will be seen that in order to obtain this grip of the concrete on the rods the latter should not be too near to the outside surface, as is also desirable in order to prevent the concrete below the rods becoming detached—particularly if subjected to heat. It is therefore desirable that the centre of the rods should be kept at least 1½ inches from the outside surface, but in shallow beams this distance cannot be obtained.

It is clear that the limit of usefulness of steel for the purpose of reinforcing concrete is reached when its elastic limit is attained, consequently in applying a factor of safety, it is more reasonable to apply it to the elastic limit of the steel than to its breaking weight. This would imply that a reinforced concrete beam should be so designed that the concrete would be on the point of failing when the elastic limit of the steel is reached, so that the factor of safety is applied to the elastic limit of the

and to the crushing strength of the concrete. By limiting the tensile stress in the steel in this manner, the strains in the concrete adhering to it are not injurious. The modulus of elasticity of the concrete is not constant, but decreases as the stress increases, as already pointed out, but as it is actually only compressed to a fraction, say one-quarter of its ultimate strength, the variation over this range is not considerable. No advantage is gained by employing unnecessary refinements in a calculation where the nature of the case involves a certain latitude in the fundamental conditions, hence the neutral plane may be assumed to be at the central horizontal section of the beam, and the stress in the concrete may be taken as varying uniformly from zero at this position to a maximum at the upper surfaces.

In the case of a large work to be constructed of reinforced concrete it would be worth while finding the modulus of elasticity of the concrete to be actually used, for the range of stress it will be subjected to. Then the true position of the neutral axis could be found by plotting the actual strains.

Diameter of Steel Rods for proper adhesion to the Concrete

It is clear that the tension in steel rods of constant cross-section at any section cannot be greater than the total adhesion of the concrete to it from the section when the stress in it is zero; and if there is a stress at the end of a rod, as in the case of rods curved upwards to take the tension due to the shear, already referred to, the ends must be satisfactorily anchored. It is found that a safe value for the force of adhesion of steel bars to concrete may be taken as $\frac{1}{10}$ ton per square inch; with a factor of safety of 4, this gives $\frac{1}{40}$ ton per square inch. There does not appear to be any advantage in using steel of high tensile strength, because although the modulus of elasticity for such steel is greater than that for mild steel, a greater stretch would be required to develop its elastic strength, which might cause visible cracks in the concrete. If the elastic limit of the steel employed be taken as 20 tons per square inch, this will be a greater value than that of the actual elastic limit of mild steel, but the concrete must necessarily take up some of the tension; consequently, the actual stress in the rods will not be as high as appears. Applying the factor of safety of 4, this gives a working intensity of tensile stress in the steel of 5 tons per square inch—the working stress is, however, often taken higher than

this. A safe value for the crushing strength of concrete of suitable strength would be 1 ton per square inch, which, with a factor of safety of 4, gives a working intensity of compression in it of $\frac{1}{4}$ ton per square inch. The rods must be kept small enough in diameter to have sufficient surface to give the necessary adhesion. Call L the length of the span and d the diameter of the rods. Then the adhesion of the round rods to the concrete per unit length equals $\pi d \times \frac{1}{40}$, which must be equal to the maximum rate of increase of the tensile stress in the rods per unit length, an expression for which we now proceed to determine. The bending moment at any section, M, equals the moment of resistance at that section, which is the moment of resistance of the concrete in compression plus the moment of resistance of the steel rods in tension. Assuming that the steel rods take all the tension, the two latter are equal. That is, the moment of resistance at any section, about the neutral axis, of the steel rods in tension equals the moment of resistance of the concrete in compression, and each equals $\frac{M}{2}$.

Assume the beam to be uniformly loaded, and let n be the number of the steel rods, t the tension in the steel rods at any section, and y_1 the depth of the rods below the neutral axis; then

$$t \times \frac{\pi d^2}{4} \times y_1 = \frac{M}{2n}.$$

Differentiating this with respect to x—

$$\frac{dt}{dx} \times \frac{\pi d^2}{4} \times y_1 = \frac{1}{2n} \frac{dM}{dx} = \frac{1}{2n} F,$$

where F is the shearing force at the section; it is a maximum at the end of the beam and there equals $\frac{wL}{2}$. Thus the maximum rate of increase of tension in one rod

$$= \frac{dt}{dx} \times \frac{\pi d^2}{4} = \frac{F}{2ny_1} = \frac{wL}{4ny_1}.$$

Now, by assumption, the tension in the steel rods at the centre equals 5 tons per square inch, therefore

$$\frac{\pi d^2}{4} \times 5 \times y_1 = \frac{1}{2n} \frac{wL^2}{8}; \therefore \frac{wL}{4ny_1} = 5 \frac{\pi d^2}{L}.$$

Therefore the maximum increase of tension in one steel rod per inch in length $= \frac{5\pi d^2}{L}$. And this equals the adhesion per unit length $= \frac{\pi d}{40}$—*i.e.* $\frac{5\pi d^2}{L} = \frac{\pi d}{40}$, or $d = \frac{L}{200}$.

This proves that for proper adhesion the diameter of the rods should be not greater than $\frac{L}{200}$ with the above data.

Next to find what proportion the cross-sectional area of the steel bears to that of the total cross-section. This is found by equating the tension in the rods to the compression in the concrete. If B = the total breadth of the beam, and n the number of the rods, $B = n\,b$. The tension in the rods $= n \times \frac{\pi d^2}{4} \times 5$.

The compression in the concrete

$$= \tfrac{1}{2} \times \tfrac{1}{4} \times \frac{D}{2} \times B = \frac{D\,B}{16};$$

$$\therefore\ n\,\frac{\pi d^2}{4} \times 5 = \frac{D\,B}{16}, \text{ or } n\,\frac{\pi d^2}{4} : D\,B = 1 : 80.$$

That is, the area of the steel = $1\frac{1}{4}$ per cent. of the total area of the concrete.*

Moment of Resistance at the Middle Section of the Beam

Take the centre of the steel rods as being situated at a depth of $\frac{3}{8}$ D below the neutral plane.

The moment of resistance of the section equals the moment of resistance of the steel rods plus the moment of resistance of the concrete

$$= n\,\frac{\pi d^2}{4} \times 5 \times \tfrac{3}{8}D + \tfrac{1}{2} \times \tfrac{1}{4} \times \frac{D}{2} \times B \times \tfrac{2}{3}\,\frac{D}{2}$$

$$= \frac{1}{80}\,D^2 B \times \frac{15}{8} + \frac{1}{48}\,D^2 B$$

$$= D^2 B\left(\frac{3}{128} + \frac{1}{48}\right)$$

$$= \frac{17}{384}\,D^2 B = \frac{D^2 B}{23} \text{ tons-inches.}$$

The Uniformly Distributed Load that the Beam will carry

The bending moment at the centre $= \frac{w\,L^2}{8}$; equating this to the moment of resistance, $\frac{w\,L^2}{8} = \frac{D^2 B}{23}$.

* See article, *Engineering News,* New York, 15th March, 1906, p. 290, by E. Godfrey.

The uniformly distributed load $= w \text{L} = \dfrac{8}{23} \times \dfrac{\text{D}^2 \text{B}}{\text{L}}$ tons; approximately $w \text{L} = \dfrac{\text{D}^2 \text{B}}{3 \text{L}}$ tons, where D, B, and L are in inches.

The Proper Distance apart of the Rods

The tension in the rods will be transmitted up to the neutral plane as a shear. Let the working intensity of shearing stress for the concrete be taken the same as the intensity of the adhesion of the concrete to the steel, viz. $\frac{1}{40}$ ton per square inch, and considering 1 inch length of the beam near the ends; we have

$$\text{B} \times 1 \times \tfrac{1}{40} = n \pi d \times 1 \times \tfrac{1}{40}.$$

Therefore the distance apart of the rods $= \dfrac{\text{B}}{n} = \pi d$, the circumference of the rods. If the rods are square the distance apart should be $4 d$. Since $\dfrac{n \pi d^2}{4} = \dfrac{1}{80}$ D B, or $\dfrac{\pi d^2}{4} = \dfrac{1}{80} \text{D} \dfrac{\text{B}}{n}$ $= \dfrac{1}{80} \text{D} \pi d$, it follows that $\text{D} = 20 d$.

And again, since $d = \dfrac{\text{L}}{200}$, $\text{D} = \dfrac{\text{L}}{10}$.

The Maximum Intensity of Tensile Stress at the Neutral Axis

The maximum intensity of shearing stress in the beam—*i.e.* at the neutral axis of the end section $= \frac{3}{2} \times \dfrac{\text{F}}{\text{B} \times \text{D}}$ (page 126) $= \frac{3}{2} \times \dfrac{w \text{L}}{2 \times \text{B} \times \text{D}} = \frac{3}{4} \times \dfrac{\text{D}^2 \text{B}}{3 \times \text{B} \times \text{D} \times \text{L}} = \dfrac{\text{D}}{4 \text{L}}$ tons per sq. in. $= \frac{1}{40}$ ton per square inch, since $\text{L} = 10 \text{D}$.

This is therefore the maximum intensity of tensile stress at the neutral axis, acting on a plane inclined at an angle of 45 deg. to the neutral plane, and it uniformly diminishes from the ends until at the centre it equals zero. The tensile stresses at points in vertical planes between the rods and the neutral plane would be the same as at the neutral plane. To resist these tensile stresses stirrups connected to the horizontal rods and sloping upwards beyond the neutral plane are often used, or some of the horizontal rods are curved upwards towards the ends and secured by nuts with washer plates or otherwi

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Monolithic Floor

When a floor consists of reinforced concrete slabs which are monolithic with the reinforced concrete beams, these slabs are more or less in the position of beams "fixed" at the ends. In the case of a beam "fixed" at the ends, the bending moment at the centre is positive and equals $\frac{w\,l^2}{24}$, and at the supports is negative and equals $\frac{w\,l^2}{12}$. This negative bending moment should therefore be allowed for by reversing the reinforcement over the supports; that is, by placing it near the top instead of near the lower surface. Also, to allow for the defect from complete fixing, a larger bending moment than $\frac{w\,l^2}{24}$ should be allowed for at the centre, though it is not necessary to allow for the full $\frac{w\,l^2}{8}$, the value of the bending moment if the ends were simply supported. If, therefore, a positive bending moment of $\frac{w\,l^2}{12}$ be allowed for at the centre and a negative bending moment of the same amount over the supports, the conditions would appear to be amply complied with. The change over of the rods from the lower part to the upper part should take place at about one-quarter the span from the ends, as the point of inflexion for beams fixed at the ends is near that position.

Reinforced Columns. Since the main object in using steel for reinforced concrete is to take up tension by the steel whilst the concrete takes up the compression, the most logical method of reinforcing columns would appear to be by introducing circular hoops or spirals of steel embedded in the column so as to resist the tendency for the column to increase in diameter when the load is applied. To prevent the spread between the hoops, thin longitudinal rods, equally spaced round the hoops placed inside the latter and extending from top to bottom of the column, are often used.

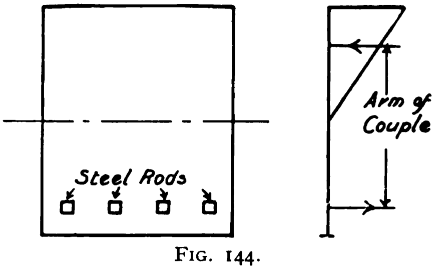

FIG. 144.

As an example of the application of the above, find what uniform load a concrete beam (Fig. 144) reinforced with four $\frac{5}{8}$-inch square steel bars will carry, its length between supports being 12 feet, its depth 12 inches, and breadth 10 inches. The centres of the rods are $1\frac{1}{2}$ inches above the under side of the beam, and $2\frac{1}{2}$ inches apart horizontally. Take the elastic limit of the steel as 18 tons per square inch, and the ultimate compressive strength of the concrete as 1 ton per square inch, and the factor of safety as 4, and assume that the neutral plane is at the centre of the depth.

The maximum compression the concrete can take in a width of $2\frac{1}{2}$ inches to limit the maximum intensity of compression in it to 1 ton per square inch $= 6 \times 2\frac{1}{2} \times \frac{1}{2} = 7\frac{1}{2}$ tons. The maximum tension that one rod can take to limit the tensile stress in it to 18 tons per square inch $= \frac{5 \times 5}{8 \times 8} \times 18 = 7$ tons. Since the latter is smaller than the former, it will be the limiting factor. The centre of the rods are at a distance of $6 - 1\frac{1}{2} = 4\frac{1}{2}$ inches below the assumed neutral plane, and the centre of resistance of the concrete is a distance above the neutral plane $= \frac{2}{3} \times 6 = 4$ inches. The arm of the resisting couple therefore equals $8\frac{1}{2}$ inches. The moment of resistance for all four rods therefore equals $\frac{7 \times 8\frac{1}{2} \times 4}{4} = 59\frac{1}{2}$ tons-inches. Equating this to the bending moment, which equals $\frac{w\,144}{8}$ tons-feet, we have $\frac{w \times 144 \times 144}{8} = 59\frac{1}{2}$, or $w = 0{\cdot}023$ tons per linear inch. $\therefore$ the total load $= w\,\text{L}$ $= 3{\cdot}3$ tons.

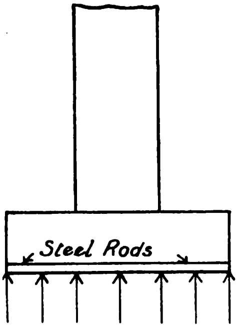

FIG. 145.

As a further example, take the case of a wall (Fig. 145) standing on a block of concrete, which projects 2 feet beyond the wall on each side, and the pressure on the foundation beneath the concrete is 2 tons per square foot. The concrete is capable of resisting a shearing stress of 60 lb. per square inch. If square steel rods $\frac{1}{2}$ inch diameter are embedded in the concrete, their centre line being 2 inches above its lower surface, and the working intensity of tensile stress in the rods is to be 5 tons per

square inch, find the proper thickness of the concrete and what distance apart the rods should be placed, also find the maximum intensity of compressive stress on a vertical section of the footing.

Call h the depth of the footing and x the distance apart of the rods in inches.

The total shear on a length of 12 inches of the footing in the vertical plane of the face of the wall equals $2 \times 1 \times 2 \times 2240$ lb., and this acts on an area of $12 \times h$ square inches; therefore the maximum intensity of shearing stress equals $1\frac{1}{2}$ times the average, *i.e.* equals $\frac{3}{2} \times \frac{2 \times 2 \times 2240}{12 \times h} = 60$ lb. per inch, the working intensity of shear; $\therefore\ h = 18\frac{2}{3}$ inches.

The tension in each rod equals $\frac{1}{2} \times \frac{1}{2} \times 5 = \frac{5}{4}$ ton.

The arm of the resisting couple equals

$$\frac{2}{3} \times \frac{h}{2} + \frac{h}{2} - 2 = \frac{5}{6}\,h - 2.$$

Therefore the moment of resistance for the length x inches equals $\frac{5}{4}\,(\frac{5}{6}\,h - 2)$ tons-inches.

The bending moment equals $\frac{x}{12} \times 2 \times 2 \times 12$ tons-inches.

Equating these—$\frac{5}{4}\,(\frac{5}{6}\,h - 2) = 4\,x,$

$$4\,x = \frac{25}{24} \times 18\frac{2}{3} - \frac{5}{2},$$

$$x = \frac{19{\cdot}5 - 2{\cdot}5}{4} = 4\frac{1}{4} \text{ inches.}$$

The maximum compressive stress f_1 in the concrete is found by equating the tension in the rods to the compression in the concrete. We have found that the former $= \frac{5}{4}$ tons, the latter $= \frac{f}{2} \times \frac{h}{2} \times x = \frac{f}{2} \times 9\frac{1}{3} \times 4\frac{1}{4} = 19{\cdot}8\,f.$

Equating the two values, $19{\cdot}8\,f = \frac{5}{4},$

$$f = \frac{1}{16} \text{ ton per square inch.}$$

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H. S.

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